MATH 1500 April 7.

We've been using the fundamental theorem of calculus, which says:

$$\frac{d}{dx}\int_c^x f(t)\,dt = f(x) \quad \text{and}$$

if $F(x)$ is an antiderivative of f, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Example: Evaluate

$$\int_0^1 \left(x\sqrt{x} + x^{-1/2}\right) dx.$$

Solution: Here, we use the fact that $\frac{x^{n+1}}{n+1}$ is an antiderivative of x^n. Then

$$\int_0^1 \left(x\sqrt{x} + x^{-1/2}\right) dx = \int_0^1 \left(x^{3/2} + x^{-1/2}\right) dx$$

$$= \left[\frac{x^{5/2}}{5/2} + \frac{x^{1/2}}{1/2}\right]_0^1$$

$$= \left[\frac{2}{5}x^{5/2} + 2x^{1/2}\right]$$

$$= \left(\frac{2}{5}(1) + 2(1)\right) - \left(0 + 0\right) = \frac{2}{5} + 2 = \frac{12}{5}.$$

Example: Evaluate $\int_1^4 \frac{x-3}{x}\,dx$.

Solution: This becomes:

$$\int_1^4 \frac{x-3}{x}\,dx = \int_1^4 \frac{x}{x} - \frac{3}{x}\,dx = \int_1^4 1 - \frac{3}{x}\,dx.$$

The formula $\frac{x^{n+1}}{n+1}$ for antiderivatives only applies for $n \neq -1$. Otherwise we get $\ln(x)$, so

$$= \Big[x - 3\ln(x)\Big]_1^4 = \Big(4 - 3\ln(4)\Big) - \Big(1 - 3\ln(1)\Big)$$

$$= 4 - 3\ln(4) - 1 + 3\cdot 0$$

$$= 3 - 3\ln(4)$$

Example: Find the area between $f(x) = x^2 - 2 + |x|$ and the x-axis, or "enclosed by f(x)" and the x-axis.

Solution: To find where the graph crosses the x-axis (ie. to find the enclosed region) we must solve $f(x)=0$, ie $x^2 - 2 + |x| = 0$.

Remember,
$$|x| = \begin{cases} x & \text{if } x>0 \\ -x & \text{if } x<0 \end{cases}$$

So the positive solutions must satisfy:

$$x^2 - 2 + x = 0$$

ie. $(x+2)(x-1) = 0$

The only positive solution to $(x+2)(x-1)=0$ is $x=1$.

The negative solutions must satisfy $x^2-2-x=0$, ie $(x-2)(x+1)=0$. The only negative solution is $x=-1$. Thus the graph crosses the x-axis at $x=\pm 1$. Between ± 1 it is below the axis, and $\lim_{x\to\pm\infty} x^2-2+|x| = \infty$, so

Thus the area we seek is

$$-\int_{-1}^{1}(x^2-2)+|x|\,dx. \quad \text{Because} \quad |x| = \begin{cases} x & \text{if } x>0 \\ -x & \text{if } x<0, \end{cases}$$

this is

$$-\int_{-1}^{1}(x^2-2)+|x|\,dx = -\left[\int_{-1}^{0} x^2-2-x\,dx + \int_{0}^{1} x^2-2+x\,dx\right]$$

$$= -\left[\frac{x^3}{3}-\frac{x^2}{2}-2x\right]_{-1}^{0} - \left[\frac{x^3}{3}+\frac{x^2}{2}-2x\right]_{0}^{1}$$

$$= -\left[(0-0-2)-\left(\frac{(-1)^3}{3}-\frac{(-1)^2}{2}-2(-1)\right)\right]-\left[\left(\frac{1}{3}+\frac{1}{2}-2\right)-(0)\right]$$

$$= +\frac{-1}{3}-\frac{1}{2}+2-\frac{1}{3}-\frac{1}{2}+2 = \frac{14}{6} = \frac{7}{3}$$

Terminology: When speaking of $\int_a^b f(x)\,dx$, where we allow some negative areas to cancel with positive, sometimes we use the terminology 'net area'.

Example: What value of $b > -1$ maximizes the net area $\int_{-1}^{b} x^2(3-x)\,dx$? What is the max area?

Solution: Such a number b will be a global maximum of the function $g(x) = \int_{-1}^{x} t^2(3-t)\,dt$ on the interval $[-1, \infty)$. To solve such a max/min problem, we need to compute $\frac{dg}{dx}$ and set it equal to zero.

By the FTC, $\frac{dg}{dx} = \frac{d}{dx}\int_{-1}^{x} t^2(3-t)\,dt) = x^2(3-x)$.

Then $\frac{dg}{dx} = 0$ gives $x^2(3-x) = 0$

$\Rightarrow x = 0$ or $x = 3$.

So the critigal points of $g(x)$ are $x=0$ and $x=3$. To test which is a max/min, we use the second derivative test:

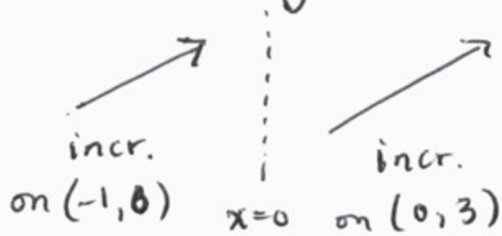
$$\frac{d^2g}{dx^2} = g''(x) = \frac{d}{dx}\left(3x^2 - x^3\right) = -3x^2 + 6x.$$

So at $x=0$ $g''(x) = 0$, which tells us nothing. At $x=3$,

$g''(3) = -3(3)^2 + 6(3) = -27 + 18$, so $g(x)$ is concave down at $x=3$ and it is a **max**.

Still we need to analyze $x=0$, since the ~~first~~ second derivative test failed there. For x in $[-1,0)$, $g'(x) = x^2(3-x)$ is positive. For x in $(0,3)$, $g'(x) = x^2(3-x)$ is again positive.

So $g(x)$ is



so

something like this.

So $g(x)$ is maximized at $x=3$. The max value is

$$\int_{-1}^{3} x^2(3-x)\,dx = \int_{-1}^{3} 3x^2 - x^3\,dx = \left[3\left(\frac{x^3}{3}\right) - \frac{x^4}{4}\right]_{-1}^{3}$$

$$= \left[x^3 - \frac{x^4}{4}\right]_{-1}^{3}$$

$$= \left(3^3 - \frac{3^4}{4}\right) - \left((-1)^3 - \frac{(-1)^4}{4}\right)$$

$$= 27 - \frac{81}{4} - (-1) + \frac{1}{4} = 8$$

Example: If $y = \int_0^x e^{-t^2+1}\,dt$,

what is the line tangent to the function $y(x)$ at $x=0$?

Solution: By the fundamental theorem of calculus I,

$\frac{dy}{dx} = \frac{d}{dx}\int_0^x e^{-t^2+1}\,dt = e^{-x^2+1}$, so the slope of

the tangent line at $x=0$ is

$$\left.\frac{dy}{dx}\right|_{x=0} = e^{-0^2+1} = e^1 = e.$$

So tangent line is $y = ex+b$, where b is chosen so that the line passes through the point $x=0$,

$y = \int_0^0 e^{-t^2+1}\,dt = 0.$ Thus $b=0$ and

$$\boxed{y = ex}.$$

MATH 1500. Last class (review).

Recall there are some required proofs! One that I did not yet cover is:

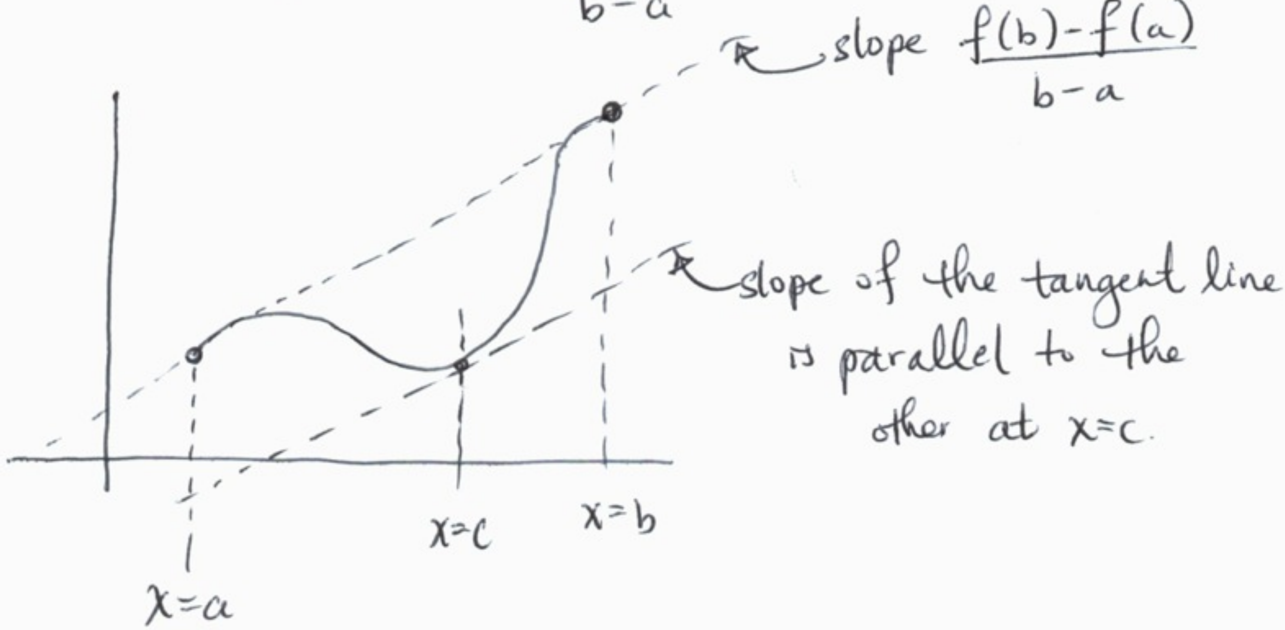
Theorem: If $f'(x) < 0$ on an interval (a,b) then $f(x)$ is decreasing on (a,b). (I did $f'(x) > 0$ $\Rightarrow$ increasing)

The proof of this requires you to know (and use) two things:

(i) A function is called decreasing if whenever $x < y$ then $f(y) < f(x)$ (it flips the order).

(ii) The Mean Value theorem says that if $f(x)$ is continuous on $[a,b]$ and differentiable on (a,b) then there is a number c in (a,b) so that

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$



Proof of theorem:

We need to start from the inequality $x_1 < x_2$ and arrive at $f(x_2) < f(x_1)$, by applying the Mean Value theorem.

So if x_1 and x_2 are numbers in (a,b) with $x_1 < x_2$, we note that $f'(x) < 0$ on (a,b) means f is differentiable and continuous on (x_1, x_2) and $[x_1, x_2]$ respectively, because $[x_1, x_2]$ is contained in (a,b). So we apply the MVT to $[x_1, x_2]$ and get c with

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

or $f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$.

Since $f'(c) < 0$ and $(x_2 - x_1) > 0$, the right hand side is negative. So

$$f(x_2) - f(x_1) < 0$$

$$\Rightarrow f(x_2) < f(x_1),$$

which is what we needed to show.

Also this kind of question:

Example: Find the value of c that makes the function continuous:

$$f(x) = \begin{cases} 9x+9 & \text{if } x \le 4 \\ -4x+c & \text{if } x > 4. \end{cases}$$

Solution:

A function is continuous at $x=a$ if $\lim_{x\to a} f(x) = f(a)$, so this is what we need to check at $a=4$.

In order for the limit to exist, we need

$$\lim_{x\to 4^-} f(x) = \lim_{x\to 4^+} f(x).$$

So $\lim_{x\to 4^-} f(x) = \lim_{x\to 4^-} 9x+9 = 9(4)+9 = 45.$

and $\lim_{x\to 4^+} f(x) = \lim_{x\to 4^+} -4x+c = -16+c$

So we need $-16+c = 45 \Rightarrow c=61.$

So if $\lim_{x\to 4} f(x)$ is going to exist, $c=61$ is required, and this makes the limit equal 45.

But more than just existing is required! We specifically need it to equal $f(4)$, which is

$f(4) = 9(4) + 9 = 45$, so

$\lim_{x \to 4} f(x) = f(4)$, and it's continuous.

[Specific warnings: Never, ever rely on Yahoo questions and be careful of other sites! Eg. integralCalc.com is terrible!].

Another question that went badly:

Example: Show that $f(x) = |x-1|$ is not differentiable at $x = 1$.

Solution: This means we must show that the number $f'(1)$ doesn't exist. The formula is

$$f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$, so we need to show this limit does not exist.

$$= \lim_{h \to 0} \frac{|1+h-1| - |1-1|}{h}$$

$$= \lim_{h \to 0} \frac{|h|}{h}.$$

So we do

since $|h| = h$ if $h > 0$

$$\lim_{h \to 0^+} \frac{|h|}{h} = \lim_{h \to 0^+} \frac{h}{h} = 1$$

$$\lim_{h \to 0^-} \frac{|h|}{h} = \lim_{h \to 0^-} \frac{-h}{h} = 1,$$ since $|h| = -h$ if $h < 0$.

Example: If $f(x) = \sqrt{3x+1}$, calculate $f'(x)$ from the definition.

Solution: The definition is

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{3(x+h)+1} - \sqrt{3x+1}}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{3x+3h+1} - \sqrt{3x+1}}{h} \cdot \frac{\sqrt{3x+3h+1} + \sqrt{3x+1}}{\sqrt{3x+3h+1} + \sqrt{3x+1}}$$

$$= \lim_{h\to 0} \frac{3x+3h+1-3x-1}{h\left(\sqrt{3x+3h+1} + \sqrt{3x+1}\right)}$$

$$= \lim_{h\to 0} \frac{3\cancel{h}}{\cancel{h}\left(\sqrt{3x+3h+1} + \sqrt{3x+1}\right)}$$

$$= \frac{3}{\sqrt{3x+1} + \sqrt{3x+1}} = \frac{3}{2\sqrt{3x+1}}.$$

A couple remarks about marking:

- If you leave off $\lim\limits_{x\to a}$ anywhere, it's -1 pt.
- If you write $\lim\limits_{x\to a}$ after plugging in $x=a$, it's -1 pt.