MATH 1500

Monday Jan 6.

Lecture 1

MWF 3:30 - 4:20.
Wallace 221.

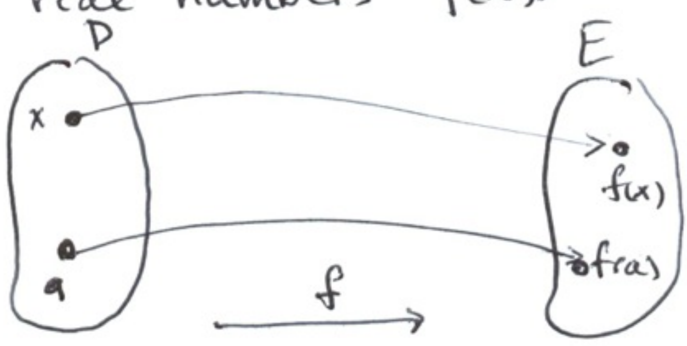
§ 1.1 Functions

A function is a rule for creating an output whenever you're given an input, functions are used to model situations where one variable/quantity depends on another.

Example: If A is the area of a circle of radius r, then there is a formula $A = \pi r^2$. We say that the area is a function of the radius, and write $A(r) = \pi r^2$.

> In general, a function f is a rule that takes in an element x from a set D, and gives back an element $f(x)$ from a set E.

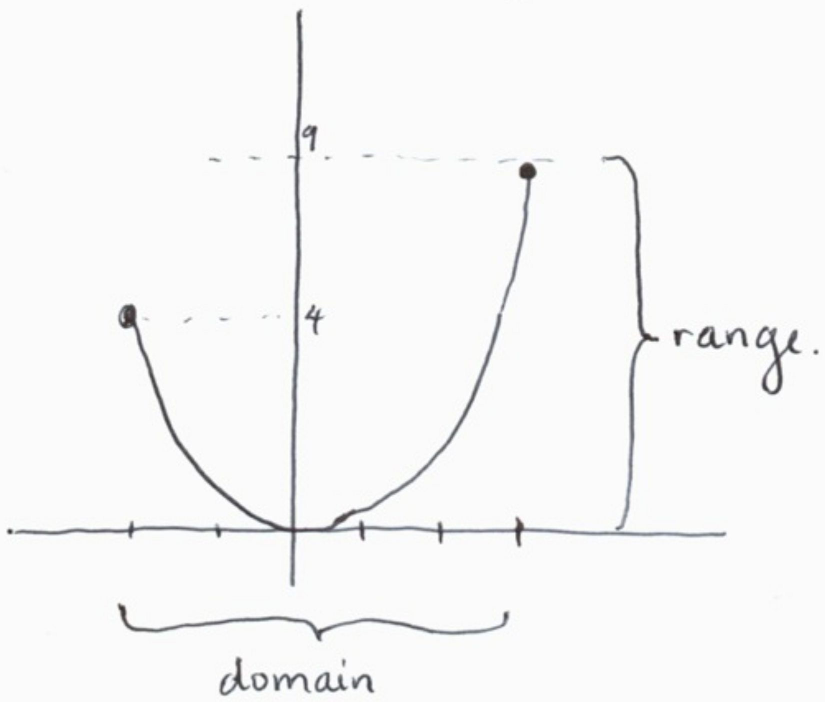
In our class, we'll pretty much exclusively talk about functions that take in real numbers x and give back real numbers $f(x)$.



Terminology:

- If f takes in elements from D, then D is called domain of f.
- The set of all possible outputs $f(x)$, as x varies over all possible elements in D, is the range of f.
- A variable representing an element of the domain of f is called an independent variable.
- A variable representing an element of the range is called dependent.

Example: Given a function like $f(x) = x^2$, we can graph it. Say we only allow x between -2 and 3. Then we get:



We would write:

The ~~range~~ domain of f is $[-2, 3]$ or $x \in [-2,3]$ or x in $[-2,3]$

Then range of f is $[0,9]$ or $y \in [0,9]$ or y in $[0,9]$.

Example: If $f(x) = x^3$, evaluate (when $h \neq 0$)

$$\frac{f(a+h) - f(a)}{h}$$

Solution: The part requiring work is $f(a+h)$. We plug in $(a+h)$ for x, and get:

$$f(a+h) = (a+h)^3 = (a+h)(a+h)(a+h)$$
$$= (a^2+2ah+h^2)(a+h)$$
$$= a^3 + 3a^2h + 3ah^2 + h^3$$

So then $\frac{f(a+h) - f(a)}{h}$ is:

$$\frac{\cancel{a^3} + 3a^2h + 3ah^2 + h^3 - \underbrace{a^3}_{= f(a)}}{h}$$

$$= \frac{3a^2h + 3ah^2 + h^3}{h} = h^2 + 3ah + 3a^2.$$

Note: This quantity $\frac{f(a+h) - f(a)}{h}$ is called a difference quotient and we will all come to love (hate) it. ☺

Example: What is the domain of

$$f(x) = \frac{x+1}{x-2}\ ?$$

Solution: In other words, what numbers can we "legally" plug in for x? Certain operations are illegal, like division by 0 or the square root of negative numbers.

Here we need $x-2 \neq 0$ to aviod division by 0.

i.e. the domain is all x, but $x \neq 2$. I.e. $(-\infty, 2) \cup (2, \infty)$.

Example: What is the domain of $f(x) = \sqrt{x-1}$?

Solution: Here we need $x-1 \geq 0$, so

$x \geq 1$ is what we allow.

I.e. the domain is $x \geq 1$ or $[1, \infty)$.

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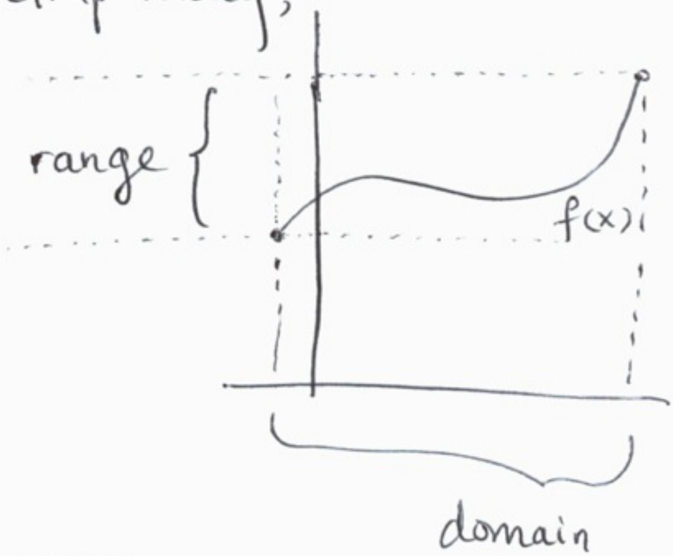
Lecture 2.

<u>Last day</u>

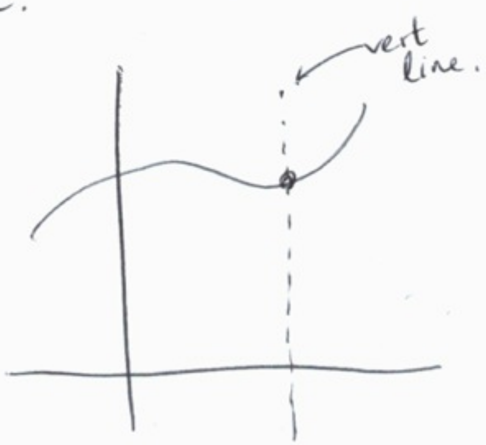
- A function f is a rule taking inputs x and giving outputs $f(x)$.
- The set of all possible inputs is the <u>domain</u>
- The set of all outputs is the <u>range</u>.

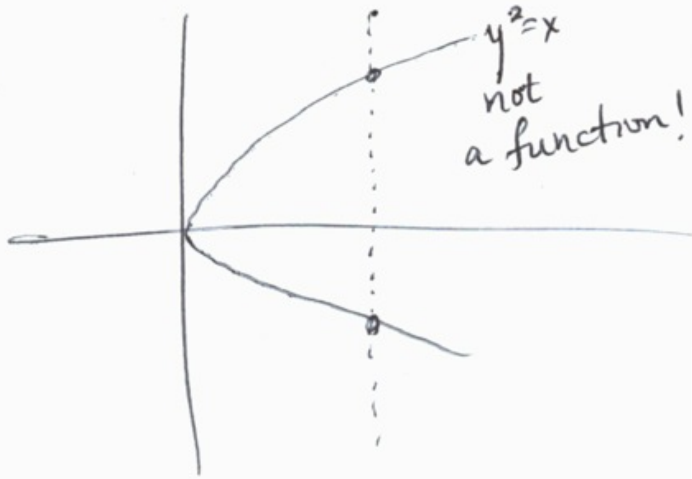
Graphically,



A curve in the plane is the graph of a function if and only if no vertical line hits it more than once.

<u>E.g</u>:





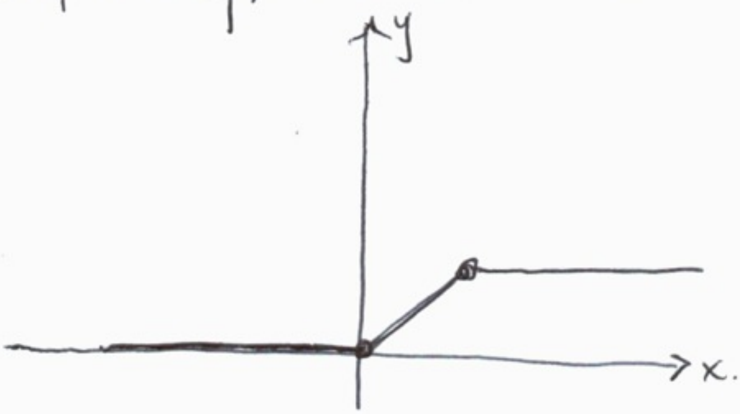
Functions can have different special properties:

① A function is 'piecewise defined' if its formula is written one segment at a time.

Example

$$f(x) = \begin{cases} 0 & \text{if } x \le 0 \\ x & \text{if } 0 < x \le 1 \\ 1 & \text{if } x > 1. \end{cases}$$

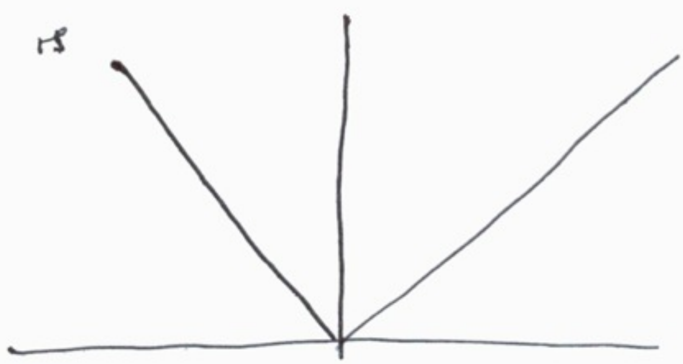
Graphically, this looks like:



An important piecewise defined function is the absolute value function, which makes negative numbers positive.

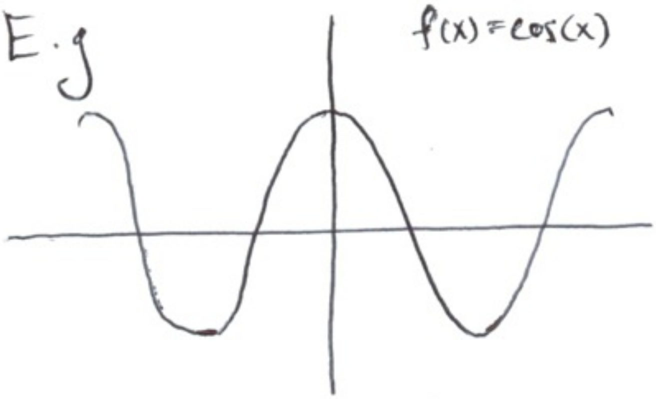
$$|x| = \begin{cases} x & \text{if } x \ge 0 \\ -x & \text{if } x < 0. \end{cases}$$

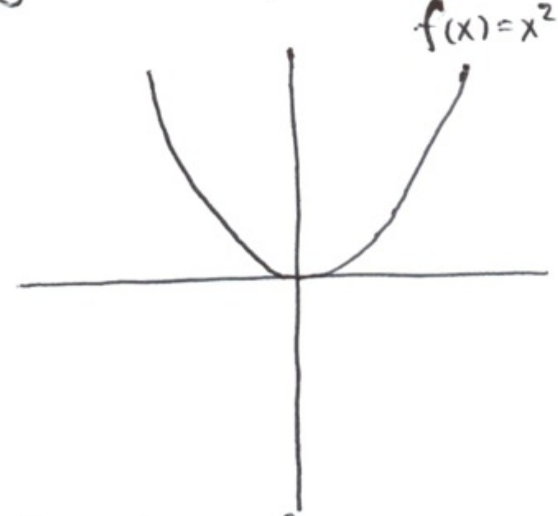
Its graph is



② A function f is <u>even</u> if its graph looks the same after reflecting in the y axis:

E.g

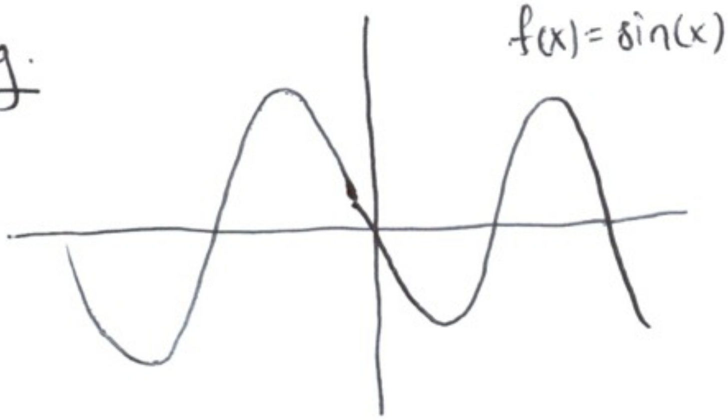


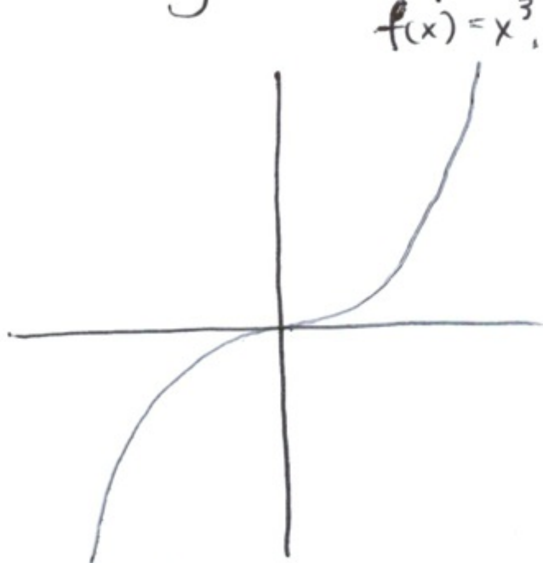


This is expressed in formulas by the eqn $f(-x) = f(x)$. (Plugging in neg or pos gives same).

③ A function f is called <u>odd</u> if the graph looks the same after reflecting in the y-axis, and then the x-axis:

E.g.





In equations, a function is odd if $f(-x) = -f(x)$ (Can take out a minus sign).

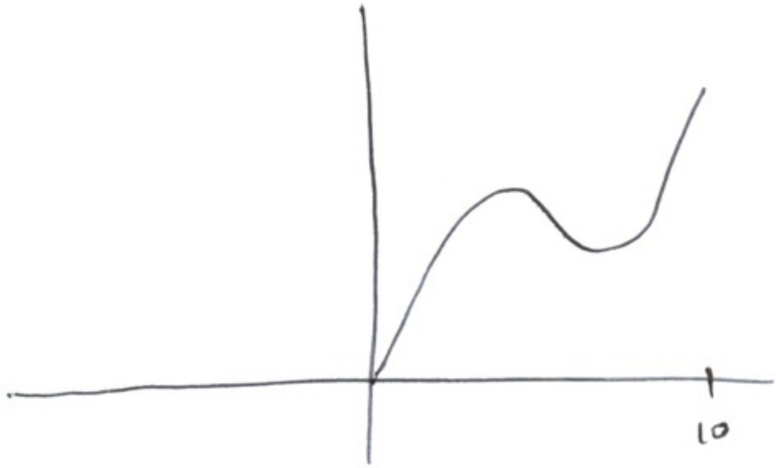
(4) A function is called increasing if its graph goes up from left to right

In equations, $f(a) < f(b)$ whenever $a < b$

and decreasing if its graph goes down from left to right.

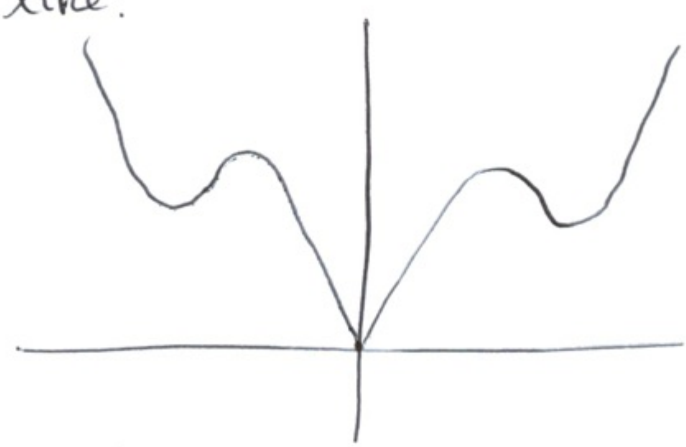
In equations, $f(a) > f(b)$ whenever $a < b$.

Example: Suppose that f is a function with domain $[-10, 10]$, and the graph of f for the part $[0, 10]$ looks like:

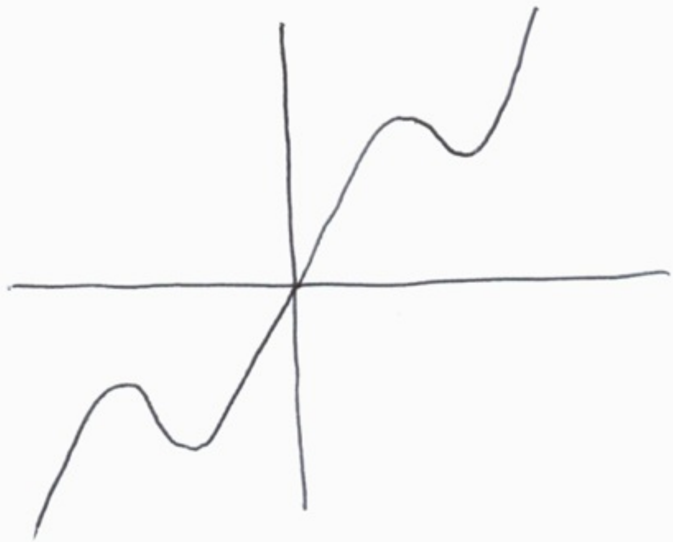


What does f look like if f is even? Odd?

Solution: If f is even then the graph doesn't change when reflecting in the y axis, so it looks like:



If it is odd, then it doesn't change when you reflect in the y axis, then x. So the graph is:



Remark: In this last example, f was neither increasing nor decreasing.

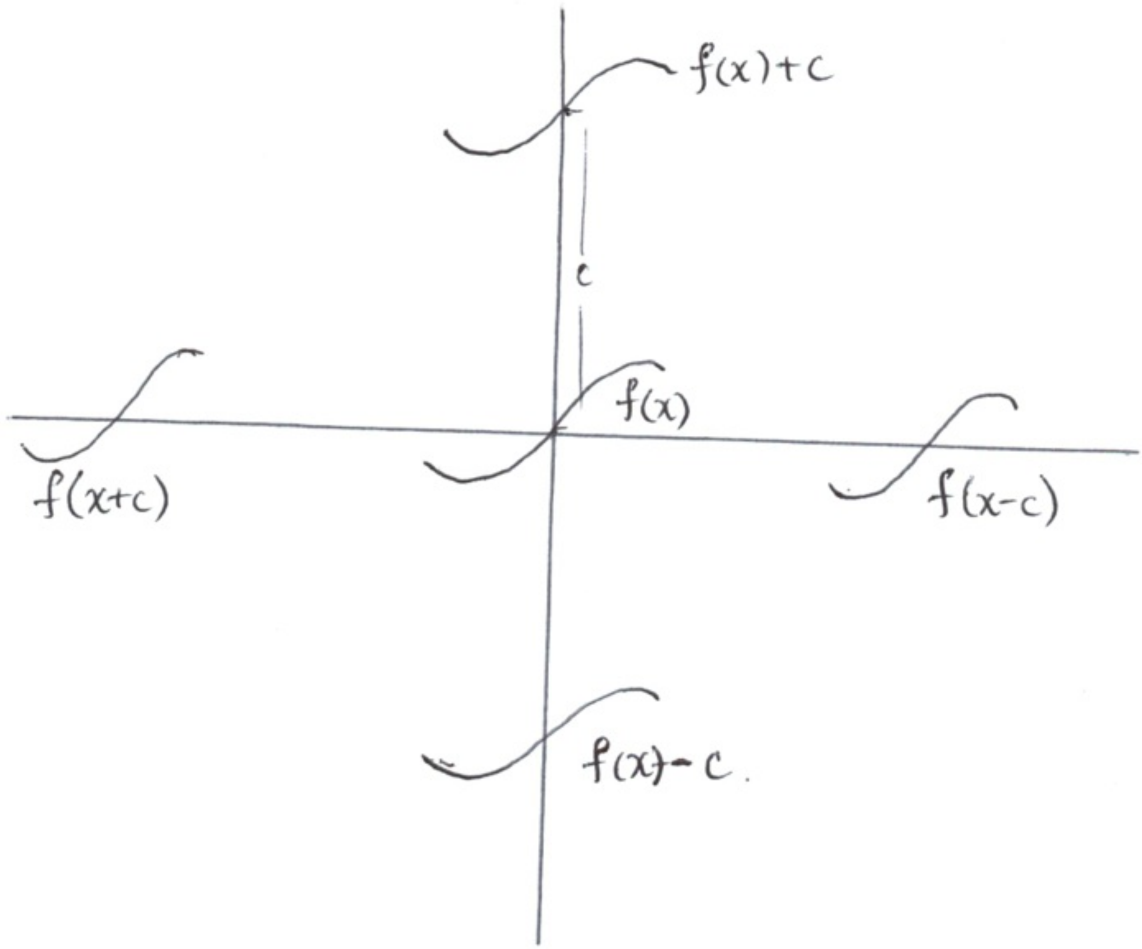
Example: Draw the graph of a function which is neither even, odd, increasing or decreasing.

§1.3 New functions from old functions.

There are several ways of transforming functions: shifting, stretching, reflecting, adding them, multiplying them, composing them.

① Shifting.

Suppose $c > 0$. Then we can shift the graph of $f(x)$ horizontally or vertically by adding/subtracting c in the right place.



In other words, to shift right we subtract (c) <u>inside</u> the brackets, shifting down we subtract c <u>outside</u> brackets.

<u>Stretching</u>

In order to <u>stretch</u> the graph of f, we multiply or divide by ~~a~~ 'c' in the right place.

In this case the graphs would all overlap if we draw them, so we give rules.

① The graph of $cf(x)$ is the same as the graph of $f(x)$, stretched vertically by a factor of c.

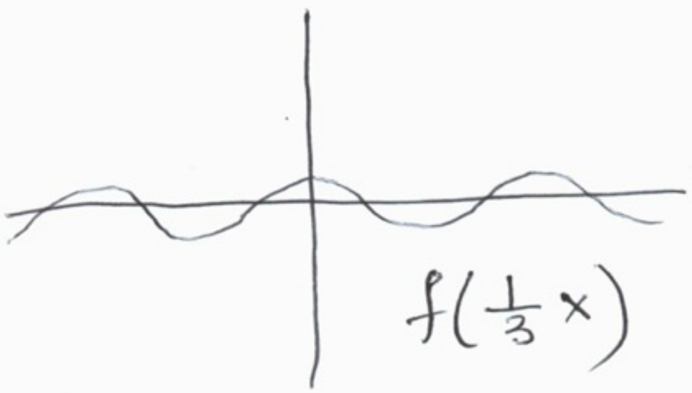
②. The graph $\frac{1}{c}f(x)$ is the same as $f(x)$, but compressed vertically by a factor of c.

③ The graph of $f(cx)$ is the same as $f(x)$, compressed horizontally by a factor of c.

④ The graph of $f(\frac{1}{c}x)$ is the same as $f(x)$, stretched horizontally by a factor of c.

End

E.g:



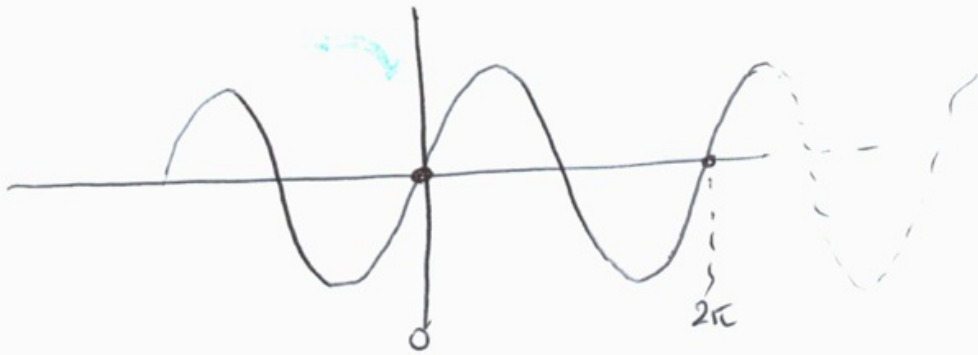
, etc.

<u>Example</u>: How often does the function

$$f(x) = \sin(5x)$$

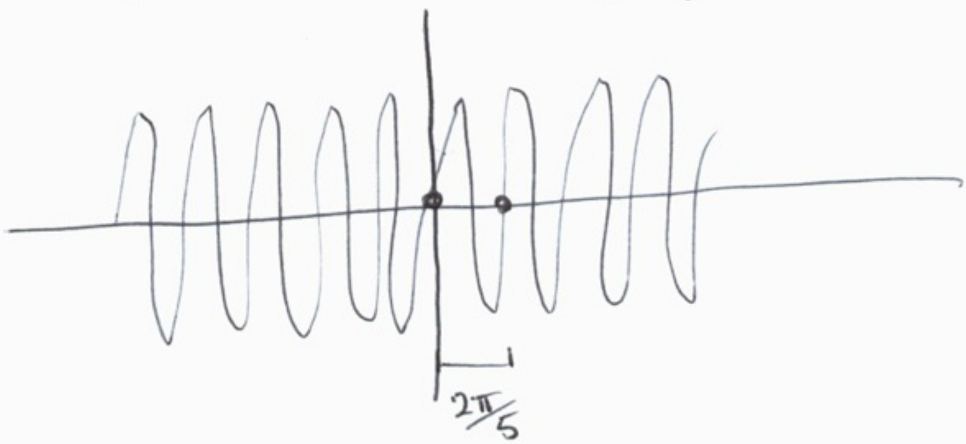
repeat ?

<u>Solution</u>: The function $\sin(x)$ has a familar graph:



which repeats every 2π.

By rule ③, $\sin(5x)$ has the same graph as $\sin(x)$, but compressed horizontally by a factor of 5:



and it now repeats every $2\pi/5$.

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Lecture 3.

- We can also reflect a function in the y-axis, by replacing $f(x)$ with $f(-x)$.
- We reflect in the x-axis by replacing $f(x)$ with $-f(x)$.

We can also make combinations of several functions, by adding them or multiplying them, or <u>composing</u> them.

If $f(x)$ and $g(x)$ are functions, then the <u>composition of f and g</u> is written

$$(f\circ g)(x) \quad \text{or} \quad f(g(x))$$

and it means that you first do the function g, then f.

<u>Example</u>: If $f(x) = \frac{x+1}{x-2}$ and $g(x) = \sqrt{x+1}$, what is the domain of $(g\circ f)(x)$?

<u>Solution</u>: We calculate

$g(f(x)) = g\left(\frac{x+1}{x-2}\right) = \sqrt{\frac{x+1}{x-2}}$. Now we need to make sure:

- no division by zero
- no square root of neg.

Division by zero happens if $x-2=0$. So we need $\boxed{x \neq 2}$.

Square root of neg happens if $\frac{x+1}{x-2} < 0$, so we need $\frac{x+1}{x-2} \geq 0$. This happens as long as $x+1$ and $x-2$ don't have opposite signs (have same signs).

So there are two cases:

(1) $x+1 \geq 0$ and $x-2 \geq 0$.

$x \geq -1$ and $x \geq 2$

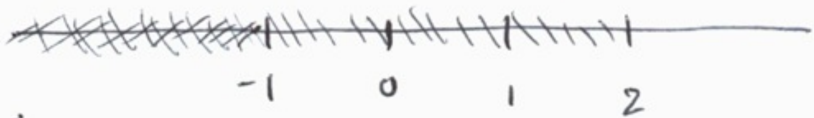
Together,

$\boxed{x \geq 2}$

(2) $x+1 \leq 0$ and $x-2 \leq 0$.

$x \leq -1$ and $x \leq 2$.

Together:



$\boxed{x \leq -1.}$

So overall, the domain of $(g \circ f)(x)$ is

$x \leq -1$ and $x > 2$.

or $(-\infty, -1] \cup (2, \infty)$.

§1.5 Exponentials review.

Recall: b^{x} — exponent.

base

An exponential function is a function where the independent variable is the exponent.

$f(x) = a^x$

Recall that if n is an integer, then

$$a^n = \underbrace{a \cdot a \cdot a \cdot a \cdot \cdots a}_{n \text{ times}},$$

and if ~~of~~ p/q is any fraction, then

$$a^{p/q} = \sqrt[q]{a^p}, \text{ same as } \left(\sqrt[q]{a}\right)^p$$

The laws of exponents:

(i) $a^{x+y} = a^x \cdot a^y$

(ii) $a^{x-y} = \dfrac{a^x}{a^y}$

(iii) $(a^x)^y = a^{xy}$

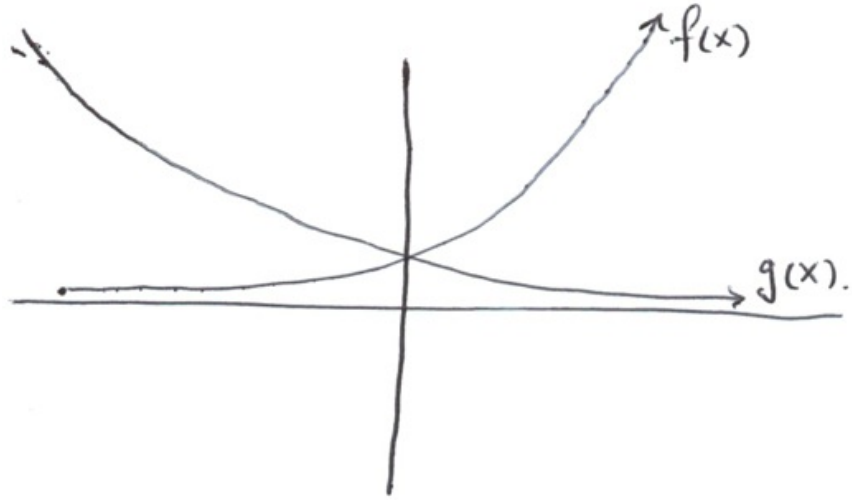
(iv) $(ab)^x = a^x b^x$.

<u>Example</u>: Sketch $f(x) = 2^x$, $g(x) = 2^{-x}$.

<u>Solution</u>: Here, $f(x) = 2^x$ obviously explodes in size as x becomes large, eg $f(10) = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdots 2 = 1024$.
The essential details of the sketch are $f(0) = 2^0 = 1$,
$f(1) = 2^1 = 2$.

Then $g(x) = 2^{-x} = \dfrac{1}{2^x}$ becomes super small, and essentially $g(0) = \dfrac{1}{2^0} = \dfrac{1}{1} = 1$, $g(1) = \dfrac{1}{2^1} = \dfrac{1}{2}$.

<u>We sketch</u>:

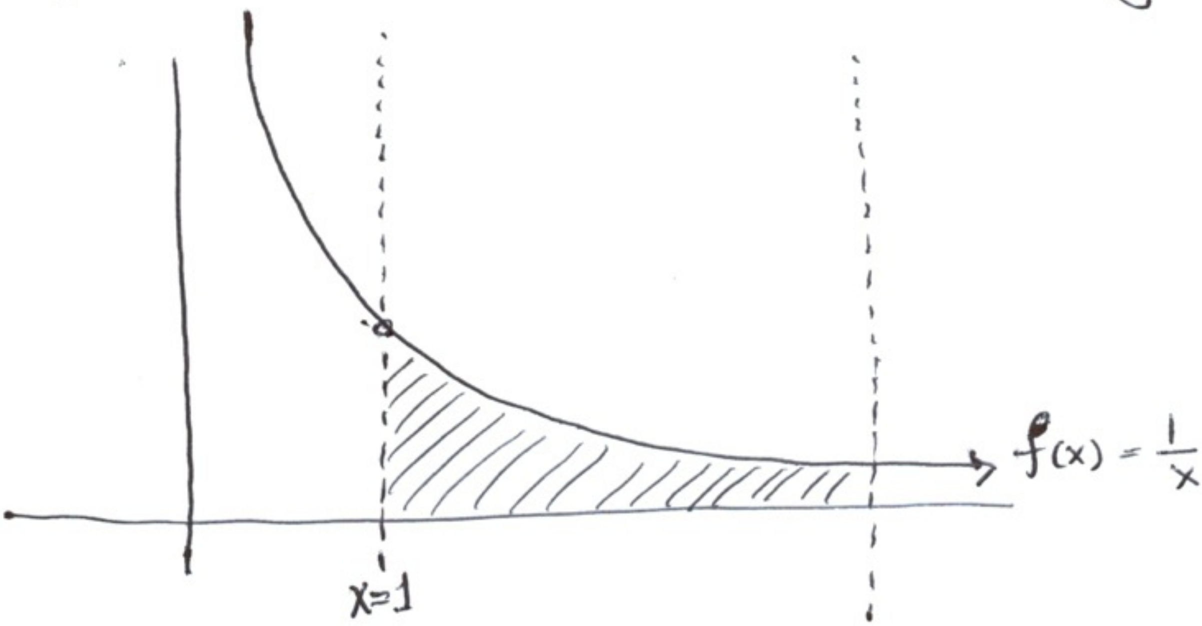


Last, we mention the most important number in calculus:

$$e \simeq 2.71828\ldots \quad \text{(continues forever, like } \pi\text{)}.$$

Our exponential functions will almost <u>always</u> use base e, ie. $f(x) = e^x$

You can also think of e this way:



Where do I have to put this line so that the area of the shaded region is exactly 1?

Answer: Put the line at $x=e$.

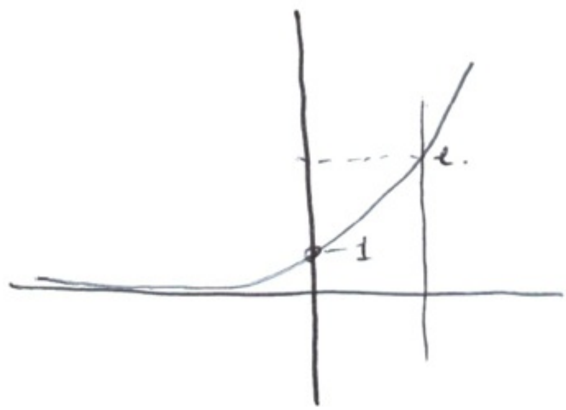
(This seems like a funny answer now, but later you will grow to love $f(x) = e^x$).

Exponential functions with base e will make life so much easier when we start calculus.

Example:

What is the range of $f(x) = 4e^x - 5$? Sketch the function.

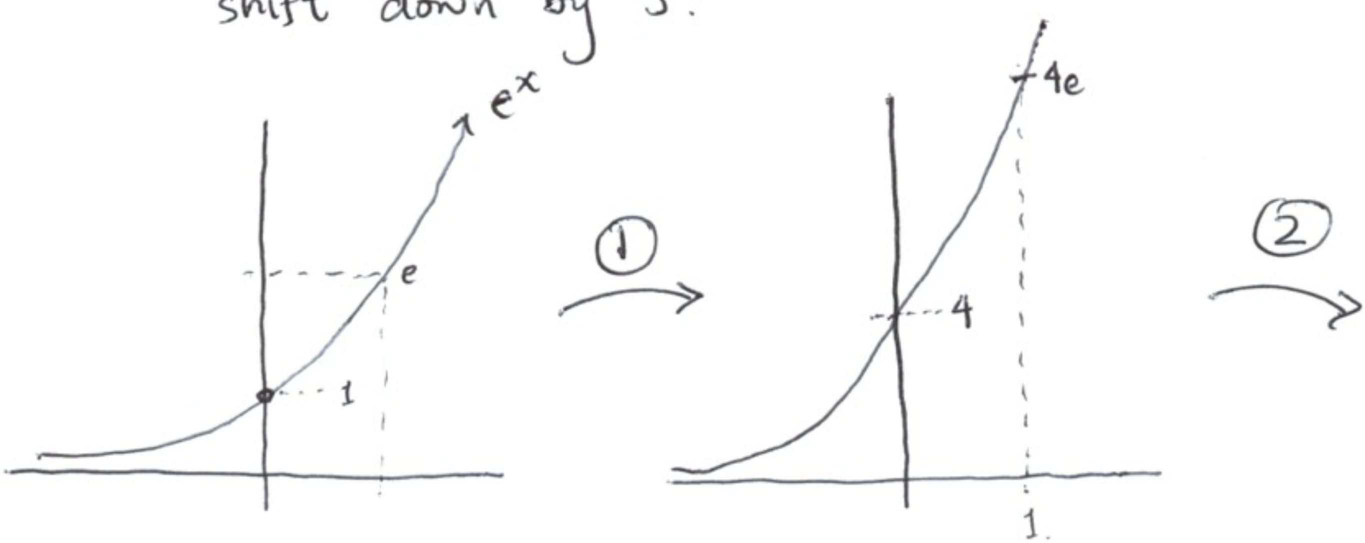
Solution: The graph of e^x is:

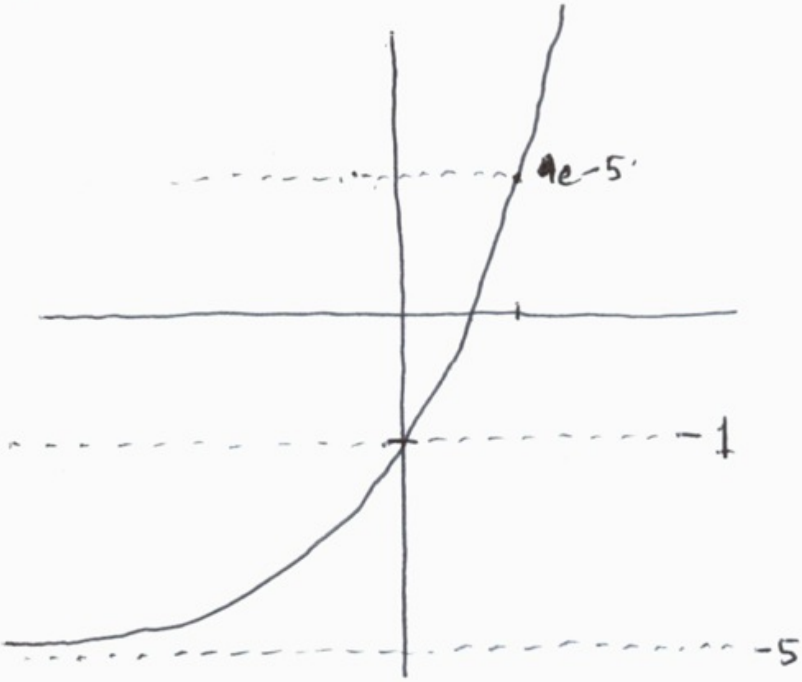


and we get from e^x to the function $f(x) = 4e^x - 5$ by applying two transformations

① First replace e^x with $4e^x$ (multiply by 4, so we stretch vertically by 4).

② Next we subtract 5 from $4e^x$, so we shift down by 5.





range is $(-5, \infty)$.

The range of $f(x)$ is $(-5, \infty)$.