Topic 9 Outline

1. Multivariable Calculus
   - Functions of Several Variables
   - 3-D Graphing
   - Partial Derivatives

Topic 9 Learning Objectives

1. define a multivariable function
2. describe differences and similarities between a function of 1 vs 2 variables
3. recognize common 3-D surfaces
4. define partial derivatives
5. calculate basic partial derivatives
6. calculate higher-order partial derivatives

What is a Multivariable Function?

Up to now, we have been working with functions of 1 variable, ie $f(x)$. The same techniques that we learned so far for functions of 1 variable can also be used for functions of more than 1 variable!

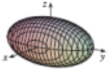
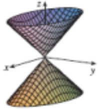
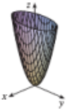
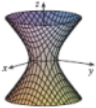
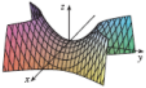
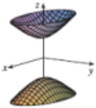
Function of 1 Variable	Function of 2 Variables

3-D Graphing

When we graph functions of 2 variables, we end up with ordered *triples* and 3-Dimensional **surfaces**.

3-D Graphing

Some common surfaces are:

Surface	Equation	Surface	Equation
Ellipsoid 	$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ All traces are ellipses. If $a = b = c$, the ellipsoid is a sphere.	Cone 	$\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces in the planes $x = k$ and $y = k$ are hyperbolas if $k \neq 0$ but are pairs of lines if $k = 0$.
Elliptic Paraboloid 	$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces are parabolas. The variable raised to the first power indicates the axis of the paraboloid.	Hyperboloid of One Sheet 	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ Horizontal traces are ellipses. Vertical traces are hyperbolas. The axis of symmetry corresponds to the variable whose coefficient is negative.
Hyperbolic Paraboloid 	$\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$ Horizontal traces are hyperbolas. Vertical traces are parabolas. The case where $c < 0$ is illustrated.	Hyperboloid of Two Sheets 	$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ Horizontal traces in $z = k$ are ellipses if $k > c$ or $k < -c$. Vertical traces are hyperbolas. The two minus signs indicate two sheets.

Partial Derivatives

Recall that for the 1-variable function $y = f(x)$, we used the notation

$$f'(x) = \frac{dy}{dx}$$

for the derivative, where $\frac{dy}{dx}$ told us that we are looking for the derivative of the function y with respect to the variable x.

For multivariable functions, we can also find the derivative, the only difference is that we now have to specify which variable we are taking the derivative with respect to! So for $z = f(x, y)$, we can take either

$$\frac{\delta z}{\delta x}, \text{or}$$

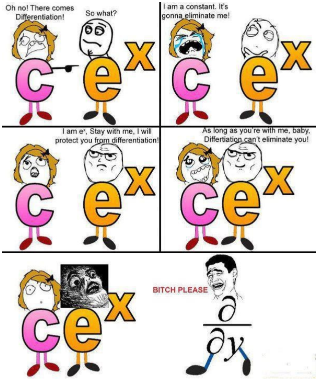
$$\frac{\delta z}{\delta y}$$

These are called **partial derivatives** because they only find the derivative with respect to *part* of the functions variables, not all.

Examples

Find $\frac{\delta z}{\delta x}$ and $\frac{\delta z}{\delta y}$ for the following:

1. $z = x^2 + y^2 + 1$
2. $z = 3xy + \frac{x}{y}$
3. $z = e^x + lny$



Higher-Order Partial Derivatives

Just as we can find $f''(x)$, $f'''(x)$, $f^{(4)}(x)$,..., we can also find higher-order partial derivatives. However, a 1-variable function has only one of each of the higher order derivatives, whereas multivariable functions have much more. How may 2nd and 3rd order derivatives does a 2-variable function have??

Examples

Find all of the 2nd order derivatives for the following:

1. $z = x^2 + y^2 + 1$
2. $z = 3xy + \frac{x}{y}$
3. $z = e^x + lny$

Five in Five!

Solve the following in 5 minutes or less!

1. Write out the equation of an exponential function of 2 variables.

2. What is the equation of a perfect sphere centred at $(0, 0)$ and with radius 3? (HINT: how is the sphere related to the ellipsoid?)

3. Find $\frac{\delta z}{\delta x}$ and $\frac{\delta z}{\delta y}$ for $z = 4x^2y + y^3 + 7$

4. Find $\frac{\delta^2 z}{\delta x \delta x}$ and $\frac{\delta^2 z}{\delta y \delta y}$ for $z = 4x^2y + y^3 + 7$

5. Find $\frac{\delta^2 z}{\delta x \delta y}$ and $\frac{\delta z}{\delta y \delta x}$ for $z = 4x^2y + y^3 + 7$

Flex the Mental Muscle!

Think of what we know about what the first and second derivatives of a 1-variable function tell us about that function's graph (increase and decrease, concavity). Can you use those same principles to discuss what the first and second *partial* derivatives would tell you about the graph of a 2-variable function?

A Preview of You and Your Final Exam