Topic 2 Outline

1. NonLinear Functions
   - What is a Function?
   - Features of Functions
   - The Exponential Function
   - Compound Interest
   - The Logarithmic Functions
   - Logarithm Laws
   - The Natural Logarithm
   - Applications of Growth and Decay
   - Effective Rate

Topic 2 Learning Objectives

1. define a function visually, numerically, and algebraically
2. sketch basic functions
3. find domain and range for various functions
4. sketch and describe piecewise defined functions
5. describe some basic features of functions
6. sketch and describe modifications of basic functions
7. describe combinations and compositions of functions
8. define and describe the exponential function
9. graph exponential functions
10. define base e for the exponential function
11. solve problems involving compound interest
12. define and graph the logarithm and natural log functions
13. recall basic facts about the logarithmic and natural log functions
14. recall and use the logarithm laws (ie, to solve equations)
15. solve problems of growth and decay, and effective rate

Four Ways to Describe a Function

A **function** is the fundamental object that we deal with in Calculus.
Functions arrise whenever one quantity depends on another.
Check out this link for a video on functions!
https://www.educreations.com/lesson/embed/9666672/?ref=app
There are four ways to describe a function:

1. verbally
2. numerically
3. visually
4. algebraically

Definition

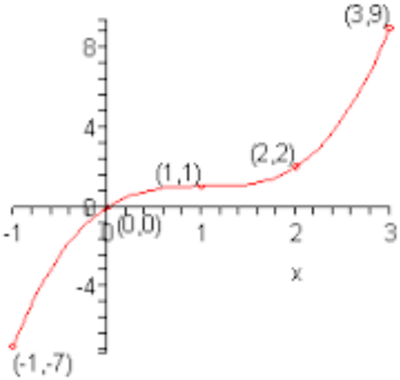
A **function** f is a rule that assigns to each element x in a set A exactly one element, called $f(x)$, in a set B.

Example

Find the values of $f(-1)$, $f(0)$, and $f(2)$ if $f(x) = x^2 - 2x + 1$.

Example

Using the graph below, find the values of $f(-1)$, $f(0)$, and $f(3)$.

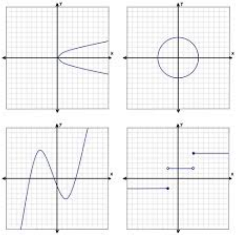


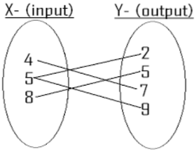
Note

Not all curves in the $x - y$ plane are the graphs of functions! A curve in the $x - y$ plane is the graph of a function $f(x)$ if and only if no vertical line intersects the curve more than once. This is the *Vertical Line Test* Why do you think this works??

Example

Which of the following are functions?





1.

2. $y = x^2$

3. $x^2 + y^2 = a^2$

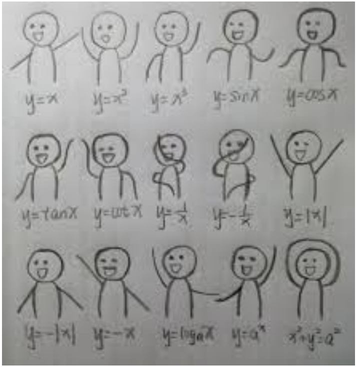
Basic Fuctions

Graph the following basic functions

1. $y = ax$
2. $y = x^2$
3. $y = x^3$
4. $y = |x|$
5. $y = \sqrt{x}$
6. $y = \frac{1}{x}$

Basic Functions

Here's a nice way to remember your basic functions!



From https://mathematicianincognito.wordpress.com

Features of a Function

We will be considering functions for which the set of inputs and set of outputs are real numbers.

- The **DOMAIN** of a function f is the set of all possible values for which $f(x)$ is defined.
- The **RANGE** of a function f is the set of all possible values of $f(x)$ as x varies through the domain.
  - The numbers/symbols in the domain are called the "independent variables".
  - The numbers/symbols in the range are called the "dependent variables".

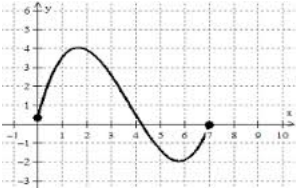
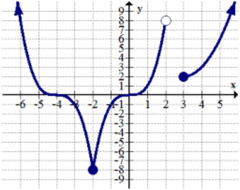
Features of a Function

Type	General	Domain	Example	Domain
Polynomial				
Rational				
Root				

Table : Types of Functions and their Domains

Example

State the domain and range (for 1-3) for the following functions:

1.  

X	Y
2	3
1	4
0	5
-1	6
-2	7

2. $f(x) = x^2$

3. $g(x) = \frac{3}{x^2-2x}$

Piecewise Defined Functions

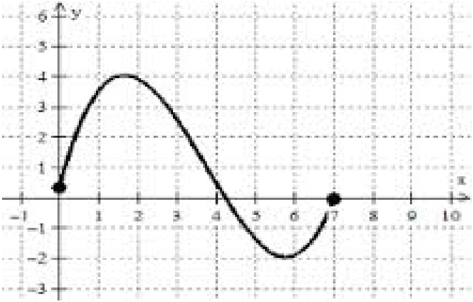
Piecewise defined functions are defined by different functions over different parts of their domain. Sketch and find the domain for the following piecewise defined functions:

- $$f(x) = \begin{cases} 2x+3 & \text{if } x < -1 \\ -x^2 & \text{if } x \geq -1 \end{cases}$$

- $f(x) = |x|$

Increasing/Decreasing

A function $f(x)$ is **increasing** on an interval I if $f(x_1) < f(x_2)$ for $x_1 < x_2$ in I. It is **decreasing** on an interval I if $f(x_1) > f(x_2)$ for $x_1 > x_2$ in I.



Symmetry

There are 2 types of symmetries that we discuss when we talk about functions:

- **EVEN** functions satisfy $f(-x) = f(x)$, and are symmetric about the y-axis.
  - ex: $f(x) = x^2$
- **ODD** functions satisfy $f(-x) = -f(x)$, and are symmetric about the origin.
  - ex: $f(x) = x^3$

New Functions from Old Functions

Once we know our basic functions, we can quickly sketch graphs and write equations for related functions by following some simple rules. If we know $y = f(x)$:

1. $y = f(x) + c \Rightarrow$ shift c units up $(c > 0)$.
2. $y = f(x) - c \Rightarrow$ shift c units down $(c > 0)$.
3. $y = f(x + c) \Rightarrow$ shift c units right $(c > 0)$.
4. $y = f(x - c) \Rightarrow$ shift c units left $(c > 0)$.
5. $y = cf(x) \Rightarrow$ stretch vertically by c $(c > 1)$.
6. $y = \frac{1}{c}f(x) \Rightarrow$ compress vertically by c $(c > 1)$.
7. $y = f(cx) \Rightarrow$ compress horizontally by c $(c > 1)$.
8. $y = f(\frac{1}{c}x) \Rightarrow$ stretch horizontally by c $(c > 1)$.
9. $y = -f(x) \Rightarrow$ reflect about the x-axis.
10. $y = f(-x) \Rightarrow$ reflect about the y-axis.

New Functions from Old Functions

We can **combine** two functions $f(x)$ and $g(x)$ to form 4 new functions:

1. $f(x) + g(x)$ or $(f + g)(x)$
2. $f(x) - g(x)$ or $(f - g)(x)$
3. $f(x)g(x)$ or $(fg)(x)$
4. $\frac{f(x)}{g(x)}$ or $(\frac{f}{g})(x)$

If the domain of f was A and the domain of g was B, then the domain of any of these new functions is the intersection of A an B.

New Functions from Old Functions

We can **compose** two functions $f(x)$ and $g(x)$ by putting one of the *inside* the other one. The notation looks like $(f \circ g)(x)$ or $f(g(x))$.

Ex: If $f(x) = x^2$ and $g(x) = \sqrt{2-x}$, find $f(g(x))$, $g(f(x))$, $f(f(x))$, and $f(g(f(x)))$

The Exponential Function

The **exponential function** is a function of the form:

$$f(x) = a^x$$

where a is a constant that is > 0, and x is a variable.
Check out this link for a video on exponential functions!
https://www.educreations.com/lesson/embed/9669753/?ref=app
To work with these functions, we must recall our rules for exponents:

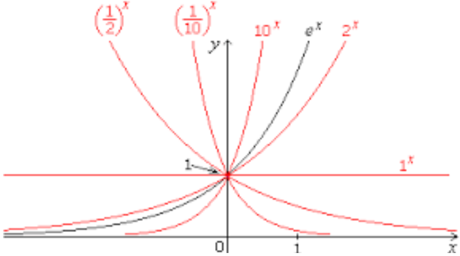
1. $a^n =$
2. $a^0 =$
3. $a^{-n} =$
4. $a^{\frac{n}{m}} =$

Example

Sketch the graphs of 2^x and $\left(\frac{1}{2}\right)^x$, and then estimate the value of $2^{\sqrt{3}}$.

The Exponential Function

Below are the graphs of some exponential functions:



Of all possible bases for an exponential function, there is one that is most convenient for calculus purposes. We call this number e, and $e \approx 2.71828...$

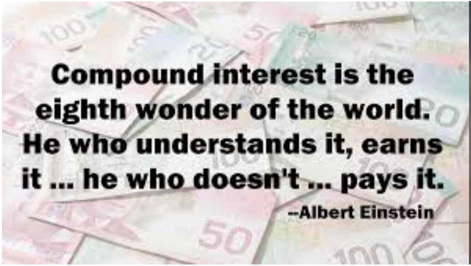
Compound Interest

Compound interest is what happens when interest is paid (or charged) on interest as well as principle (vs simple interest, where interest is ONLY paid on principle)!

If P dollars is invested at a yearly rate of interest of r per year, compounded m times per year for t years, derive the formula for Compound Interest:

Example

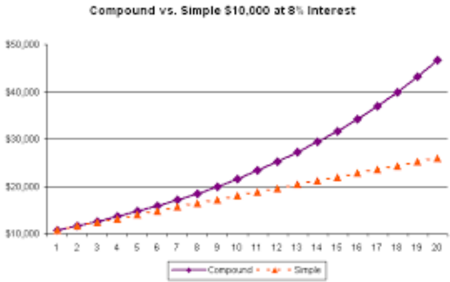
Compound interest quotes from 'influential' folks!



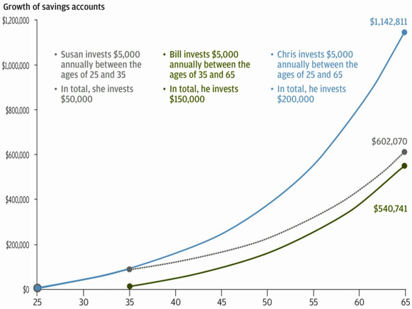
Example

If a man invests \$10,000 into an account earning 8% annual interest for 10 years, how much more will he make in an account offering quarterly compounded interest vs simple interest calculated quarterly?

Example



Compound vs. Simple $10,000 at 8% Interest
$50,000
$40,000
$30,000
$20,000
$10,000
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20
Compound
Simple



Growth of savings accounts
$1,200,000
$1,000,000
$800,000
$600,000
$400,000
$200,000
$0
25 30 35 40 45 50 55 60 65
• Susan invests $5,000 annually between the ages of 25 and 35
• In total, she invests $50,000
• Bill invests $5,000 annually between the ages of 35 and 65
• In total, he invests $150,000
• Chris invests $5,000 annually between the ages of 25 and 65
• In total, he invests $200,000
$1,142,811
$602,070
$540,741

Compound Interest

As m becomes larger and larger, the value of $(1 + \frac{1}{m})$ gets closer and closer to the number e!

This means that if a deposit of P dollars is invested at a yearly rate of interest of r per year **compounded continuously** for t years, the total amount will be:

$$A = Pe^{rt}$$

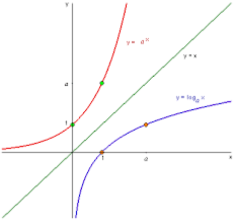
How??

Example

Assuming continuous compounding, what will it cost to buy a \$10 item in 3 years at inflation rates of 1%, 3%, and 5% ?

Logarithmic Functions

If $a > 0$ and $a \neq 1$, then the function a^x has an inverse called the **logarithmic function with base a**, $log_a(x)$.



Domain: Range:

We can transform back and forth between logarithmic and exponential functions by using the relationship:

$$log_a x = y \Leftrightarrow a^y = x$$

Logarithmic Functions

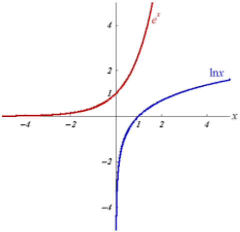
By the cancellation equations:

Log Laws:

1. $log_a(xy) = log_a x + log_a y$
2. $log_a(\frac{x}{y}) = log_a x - log_a y$
3. $log_a(x^r) = r log_a x$

The Natural Logarithm

Of all possible bases for log, the base e is the most frequently used. We call this the **natural logarithm**, $log_e x = ln x$.



Facts to remember:

Examples

1. Evaluate $log_2(80) - log_2(5) = x$
2. Solve for x in $logx = 5$
3. Solve for x in $e^{5-3x} = 10$
4. Solve for x in $2ln(4x) = 1$
5. Express $ln(1+x^2) + \frac{1}{2}lnx - ln(sinx)$ as a single logarithm
6. Find the domain of $f(x) = log(3-x)$
7. Find the domain of $f(x) = \sqrt{3 - e^{2x}}$
8. Find the domain of $f(x) = ln(2 + lnx)$

Applications in Growth and Decay

If y_0 is the amount of some quantity present at any time $t = 0$, then the amount present at any later time can be given by the **exponential growth and decay function**:

$$y = y_0 e^{kt}$$

Example: Carbon 14 is a radioactive form of carbon that is found in all living organisms. After a plant or animal dies, the C-14 disintegrates and scientists can determine the age of remains by comparing its C-14 levels to the amount found in the living organism.
The amount of C-14 present after t years is given by:

$$A = A_0 e^{kt}$$

If $k = -[\frac{\ln 2}{5600}]$ for some substance, what is its half-life?

Effective Rate

If we invest \$1 at 8% interest (per year) compounded semi-annually, we will have \$1.0816 after 1 year.
The actual amount of interest earned in that year if it had been compounded annually was 8.16% rather than the 8%.
If r is the annual nominal (stated) rate of interest and m is the number of compounding periods, then:

$$r_E = (1 + \frac{r}{m})^m - 1, \text{(for compound interest), and}$$

$$r_E = e^r - 1, \text{(for continuous compounding),}$$

Example: What is the effective rate of interest for \$1000 at 6% per year, compounded quarterly for a year, versus compounded continuously for a year?

Five in Five!

Solve the following in 5 minutes or less!

1. Find the domain of $\frac{\sqrt{x}}{4-x}$
2. What is the domain of the function $ln(\frac{x}{x-1})$?
3. Sketch a graph of the function $log_2 x$.
4. State the value of $9^{\frac{3}{2}}$
5. Sketch the piecewise defined function
$$f(x) = \begin{cases} x^3 & \text{if } x \leq 0 \\ e^x & \text{if } x > 0 \end{cases}$$

Flex the Mental Muscle!

The exponential function occurs frequently in mathematical models of nature and society, in particular, in the descriptions of population growth and decay.

The *half-life* of stronium-90 is 25 years. This means that half of any given quantity of stronium-90 will disintegrate in 90 years.

1. If a sample of stronium-90 has a mass of 24mg, find an expression for the mass $m(t)$ that remains after t years.

2. Find the mass remaining after 40 years.