Topic 5 Outline

1. Derivative Rules
   - Calculating the Derivative Using Derivative Rules
   - Marginal Cost, Revenue and Profit
   - Implicit Functions

Topic 5 Learning Objectives

1. calculate the derivative of:
    - polynomials and basic exponentials
    - products
    - quotients
    - composite functions
    - exponential and logarithmic functions
2. use logarithmic differentiation
3. calculate marginal cost, revenue, and profit
4. distinguish when and how to use each of the rules above, including combinations of them
5. calculate the derivative of implicit functions

Derivative Rules

We can find derivatives in a faster way than using the limit definition of the derivative, which can be tedious and nearly impossible even for simple functions!

Check out this link for a video on the shortcut derivative rules!
https://www.educreations.com/lesson/embed/9725600/?ref=app

Derivatives of Polynomials and an Exponential

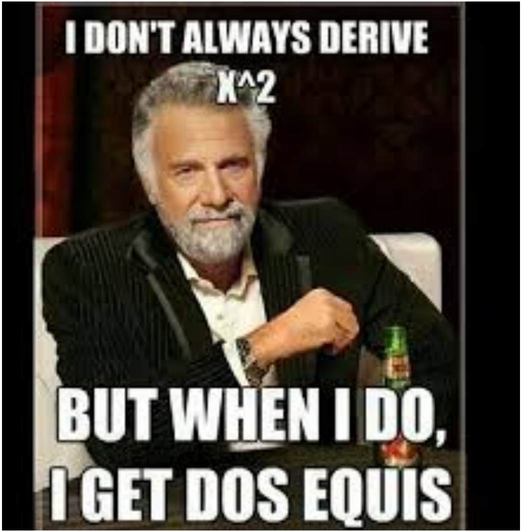
1. $\frac{d}{dx}(c) =$

2. $\frac{d}{dx}(x^n) =$ Find the derivatives of the following:
    1. $f(x) = x^6$
    2. $f(x) = x^100$
    3. $f(x) = \frac{1}{x}$
    4. $f(x) = \sqrt{x}$
    5. $f(x) = \sqrt[3]{x^4}$

3. $\frac{d}{dx}(e^x) =$

4. What is the slope of the tangent line to the curve $y = e^x$ at $x = 0$?

Derivatives of Polynomials and Exponentials



Derivative Laws

There are also two basic laws for calculating derivatives, they say that:

- $\frac{d}{dx}[cf(x)] = c\frac{d}{dx}[f(x)] = cf'(x)$

- $\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x) = f'(x) \pm g'(x)$

Examples

Find the derivatives of the following:

1. $f(x) = 186.5 + \pi$

2. $y = 3e^x + e^2$

3. $g(t) = \frac{4}{\sqrt{t}} + (\frac{1}{2}t)^5$

4. $p(r) = \frac{r^2+4r+3}{\sqrt{r}}$

5. Find the points on the curve $y = x^4 - 6x^2 + 4$ where the tangent line is horizontal.

The Product Rule

If the derivative law tells us that $[f(x) \pm g(x)]' = f'(x) \pm g'(x)$, we might also assume that $[f(x)g(x)]' = f'(x)g'(x)$.
Is this true?? Let's check using $f(x) = x$ and $g(x) = x^2$...

The Product Rule: $[f(x)g(x)]' = f'(x)g(x) + g'(x)f(x)$

The Quotient Rule

If the derivative law tells us that $[f(x) \pm g(x)]' = f'(x) \pm g'(x)$, we might also assume that $[\frac{f(x)}{g(x)}]' = \frac{f'(x)}{g'(x)}$.
Is this true?? Let's check using $f(x) = x$ and $g(x) = x^2$...

The Quotient Rule: $[\frac{f(x)}{g(x)}]' = \frac{f'(x)g(x) - g'(x)f(x)}{g(x)^2}$

Examples

Find the derivatives of the following:

1. $f(x) = xe^x$

2. $y = \frac{3x^2+2\sqrt{x}}{x}$

3. $g(t) = \frac{t^3 e^t}{3t+t^e}$

4. Suppose $f(5) = 1$, $f(5) = 1$, $f(5) = 1$ and $f(5) = 1$, find
    1. $(f+g)'(5)$
    2. $(fg)'(5)$
    3. $(\frac{g}{f})'(5)$

Marginal Cost, Revenue and Profit

We noted before that the marginal cost function is the derivative of the cost function, $C(x)$:

Marginal Cost $= C'(x)$

Marginal Revenue $= R'(x) = price(x)$

Marginal Profit $= P'(x)=[R(x) - C(x)]'=R'(x) - C'(x)$

Example

If the cost function is $C(x) = 2x^3 + x + 9$ and revenue $R(x) = 5x^3 + x$, find the average cost function, marginal average cost function, and marginal profit function. What is the marginal average profit on 10 units?

The Chain Rule

So far, we can calculate the derivatives of most functions (polynomials, sums, differences, products, quotients...). However, we have not yet seen how to find the derivative of a function that is *inside* another function - a composite function!
Examples:

So, if f and g are both differentiable and $F = f \circ g$ is the composite function defined by $F(x) = f(g(x))$, then F is differentiable and the **Chain Rule** is:

$$[f(g(x))]' = F'(x) = f'[g(x)]g'(x)$$

Examples

Differentiate:

1. $f(x) = \frac{2}{x+1}$

2. $g(t) = \sqrt{5e^x + 1}$

Derivatives of Logs and Exponentials

1. $\frac{d}{dx}(log_a g(x)) = \frac{g'(x)}{g(x)lna}$
2. $\frac{d}{dx}(log_a x) = \frac{1}{xlna}$
3. $\frac{d}{dx}(lng(x)) = \frac{g'(x)}{g(x)}$
4. $\frac{d}{dx}(lnx) = \frac{1}{x}$
5. $\frac{d}{dx}(a^{g(x)}) = a^{g(x)}g'(x)lna$
6. $\frac{d}{dx}(a^x) = a^x lna$
7. $\frac{d}{dx}(e^{g(x)}) = e^{g(x)}g'(x)$
8. $\frac{d}{dx}(e^x) = e^x$

Check out this link for a video on the log functions and their derivatives!
https://www.educreations.com/lesson/embed/9773346/?ref=app

Examples

Find $f'(x)$ if $f(x) =$

1. $ln(4x+4)$

2. $10^{x^2} + 3^x$

3. $log_{10}(\frac{x}{x-1})$

4. $\sqrt{lnx}$

5. $ln|x|$

Logarithmic Differentiation

Sometimes, the calculation of derivaties of complex functions can be made easier by taking logarithms! Steps:

1. Take natural logs (or any other base would also do) of both sides of an equation that is in the form $y = f(x)$ and use the log laws to simplify it.
2. Use implicit differentiation with respect to x to differentiate.
3. Solve for y', and this is the derivative we were looking for!

Example: Differentiate $y = \frac{x^{\frac{3}{4}}\sqrt{x^2+1}}{(3x+2)^5}$

Logarithmic Differentiation

In other cases, Logarithmic differentiation is necessary because none of the other rules will do!

This happens in cases where our function has the form $y = f(x)^{g(x)}$.

Example: Differentiate $y = 3x^{\ln x}$

Steps for Derivatives

1. Which rule(s) do I need to use?
   1. $\frac{d}{dx}(c) = 0$
   2. $\frac{d}{dx}(x^n) = nx^{n-1}$
   3. $\frac{d}{dx}(e^x) = e^x$
   4. Product Rule $(fg)' = f'g + g'f$
   5. Quotient Rule $(\frac{f}{g})' = \frac{f'g - g'f}{g^2}$
   6. Chain Rule $[f(g(x))]' = f'[g(x)]g'(x)$
   7. Rules for Exponentials and Logs $\frac{d}{dx}(log_a g(x)) = \frac{g'(x)}{g(x) lna}$, $\frac{d}{dx}(a^{g(x)}) = a^{g(x)} g'(x) lna$
   8. Logarithmic differentiation for $f(x)^{g(x)}$
2. Start with the "Big Picture" rules first, then work your way inside!

Examples

Let's put everything together to work through some more complex examples:

1. $f(x) = (\frac{x-2}{2x+1})^2$

2. $g(t) = (3t + \pi)^4(t^7 - t - 9)^5$

3. $h(t) = e^{\frac{3t}{t+1}} + ln(\frac{3t}{t+1})$

4. $p(t) = \sqrt[3]{1 + 2^t}$

Five in Five!

Solve the following in 5 minutes or less!

1. Find $f'(x)$ if $f(x) = 2x^9 + x^e + e^x + e^3$
2. Find y' if $y = ln(e^x x^3)$
3. Find $f'(x)$ if $f(x) = \frac{4x^6 - 1}{\sqrt{9+17x}}$
4. Differentiate $y = 3^x + x^3 + log_3 x$.
5. Differentiate $y = x^x$

Flex the Mental Muscle!

1. Differentiate $y = x^{x^2} + 7^{x^2}$, $x > 0$.

2. Differentiate
$$y = \frac{\sqrt[3]{x-4}(1+2x^3)^5}{\sqrt{1+x^2}}$$
once without logarithmic differentiation, and once with. Simplify your final answers until they match.