Topic 6 Outline

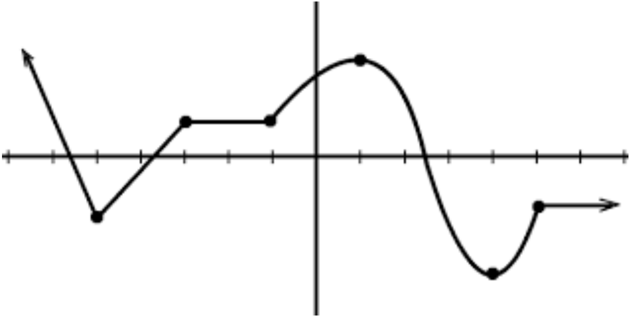
1. Curve Sketching
   - How Derivatives Affect the Shape of a Graph
   - Higher Order Derivatives
   - The Guidelines for Curve Sketching

Topic 6 Learning Objectives

1. identify the regions of increase/decrease of a function
2. prove that if $f'(x) > 0$ for all x in an interval (a, b), then f is increasing on (a, b).
3. prove that if $f'(x) < 0$ for all x in an interval (a, b), then f is decreasing on (a, b).
4. find local max and min values using the first derivative test
5. calculate higher order derivatives
6. identify the regions of concave up/concave down of a function
7. find inflection points
8. find local max and min values using the second derivative test
9. sketch curves using information from $f(x)$, $f'(x)$, and $f''(x)$

What Does the First Derivative Tell Us About the Shape of a Graph?

We already know that the first derivative can tell us about where a function is constant, but it can also tell us where a function is **increasing** or **decreasing**:



What Does the First Derivative Tell Us About the Shape of a Graph?

- If $f'(x) > 0$ for all x in an interval (a, b), then f is increasing on (a, b).

- If $f'(x) < 0$ for all x in an interval (a, b), then f is decreasing on (a, b).

Examples

Where are the following functions increasing and decreasing?

1. $f(x) = 2x^3$

2. $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$

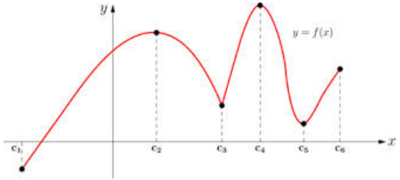
3. $g(x) = x + 2e^x$

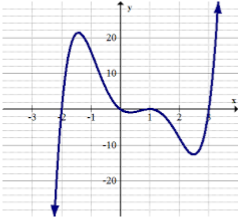
Local Extrema

What happens when a function changes from increasing to decreasing or vice-versa?

A function $f(x)$ has a **local (relative) max** at c if $f(c) \geq f(x)$ when x is near c. Similarly, $f(x)$ has a **local (relative) min** at c if $f(c) \leq f(x)$ when x is near c

Example: Identify the local extrema in the graphs below:





Local Extrema

How do we find local max's and min's??

Fermat's Theorem: If $f(x)$ has a local max or min at $x = c$, and if $f(c)$ exists, then $f'(c) = 0$. BUT, we can not expect to locate extreme values simply by setting $f'(x) = 0$ and solving.

Why is that??

So Fermat's Theorem suggests that we should at least start looking for extreme values of f at the numbers $x = c$ where $f'(c) = 0$ or $f'(c) dne$! We call these numbers (c's) the **critical numbers** or **critical points** (cp's) of the function.

Examples

Find the critical numbers of the following functions:

1. $f(x) = x^3 + +x^2 + x$

2. $f(x) = x^{\frac{3}{5}}(4 - x)$

The First Derivative Test

- If $f'(x)$ changes from $+$ to - at c, then c is a local max (as long as it is in the domain).
- If $f'(x)$ changes from - to $+$ at c, then c is a local min (as long as it is in the domain).
- If $f'(x)$ does not change sign at c, then there is no local max or min at c.

Use this test to identify the local max's and min's in the last example.

Higher Order Derivatives

If f(x) is a differentiable function, then its derivative is also a function, and so may have derivatives of its own!

For example, we know that the derivative of a function can tell us about whether that function is increasing or decreasing. If we are interested in a product whose profit function is $P(x)$, and if we know that $P'(x) > 0$, then we know that profit is always increasing. Howver, whether or not this is a good investment depends also on the *rate of increase*. This rate is $P''(x)$.

Example: Given the profit functions below (t is in years), which of the following products should I invest in for a long term investment? short term investment?

$$P_1(x) = x^3, \quad P_2(x) = x^3 - 6x^2 + 12x$$

Examples

Other notations:

1. For $f(x) = x^5 + 3^4 + 9x^3 - x^2 - x + 7$, find $f'''(x)$ and $f^{(10)}(x)$.

2. Find $\frac{d^{27}}{dx^{27}}(xe^x)$

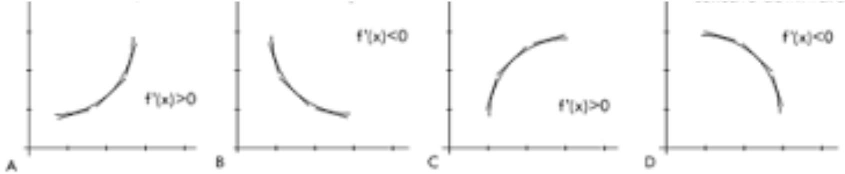
Higher Order Derivatives

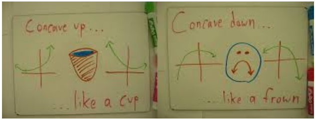
In general, we can interpret the second derivative as a rate of change of a rate of change. If the first derivative is *velocity*, then its derivate (the second derivative) represents *acceleration*.

Example: The position of a particle is given by the equation of motion $s = f(t) = t^3 - 6t^2 + 9t$ (where t is in seconds and s is in meters). Find the accelearation at time t. What is the acceleration after 4 seconds? Graph all three functions together and discuss the particles *speed*.

What Does the Second Derivative Tell Us About the Shape of a Graph?

Look at the following functions:

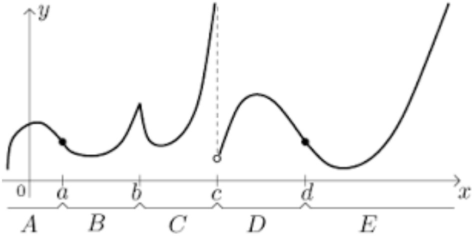




What Does the Second Derivative Tell Us About the Shape of a Graph?

- If $f''(x) > 0$ for all x in an interval (a, b), then f is concave up on (a, b).
- If $f''(x) < 0$ for all x in an interval (a, b), then f is concave down on (a, b).

The places where f changes from one concavity to the other are called **inflection points**.



Example

Sketch a possible graph of a function that would satisfy the following:

1. $f'(x) > 0$ on $(-\infty, 1)$ and $f'(x) < 0$ on $(1, \infty)$,
2. $f''(x) > 0$ on $(-\infty, 2)$ and $(2, \infty)$, and $f''(x) < 0$ on $(-2, 2)$,
3. $\lim_{x\to-\infty} f(x) = -2$ and $\lim_{x\to\infty} f(x) = 0$

The Second Derivative Test

The second derivative can also help us to distinguish between local max's and min's:

- If $f'(c) = 0$ and $f''(c) > 0$, then f has a local ______ at c.
- If $f'(c) = 0$ and $f''(c) > 0$, then f has a local ______ at c.
- If $f'(c) = 0$ and $f''(c) = 0$, then the test is inconclusive.

Example

Discuss the curve $y = x^4 - 4x^3$ with respect to regions of increase/decrease, local maxima and minima, concavity, points of inflection. Use this info to conjecture a graph for the function.

The Guidelines For Curve Sketching

To accurately sketch the graph of a function $f(x)$, follow the steps:

1. Domain: check where $f(x)$ is undefined.
2. Intercepts: x-intercept $\to$ set $y = 0$, and y-intercept $\to$ set $x = 0$.
3. Symmetry: check if $f(-x) = f(x)$ (even)$f(-x) = -f(x)$ (odd).
4. Asymptotes: HA $\to$ check $\lim_{x \to \pm\infty} f(x)$ for $y = HA$. VA $\to$ check $\lim_{x \to a} f(x)$ at all places a where $f(x)$ is undefined for $x = VA$.
5. Inc/Dec: compute $f'(x)$ and find the intervals where $f'(x) > 0$ (inc) and $f'(x) < 0$ (dec), and critical numbers where $f'(x) = 0$ or $f'(x)$ dne.
6. Local Max/Mins: check your critical numbers to see if they are local maxs or mins.
7. Concavity: compute $f''(x)$ and find *possible* inflection points, then find the intervals where $f''(x) > 0$ (CU) and $f''(x) < 0$ (CD), and confirm inflection points.
8. Sketch: put the info from the steps together to make a sketch, labeling important points first!

Examples

Check out this link for a video on curve sketching!
https://www.educreations.com/lesson/embed/9874572/?ref=app
Sketch the following functions using the guidelines:

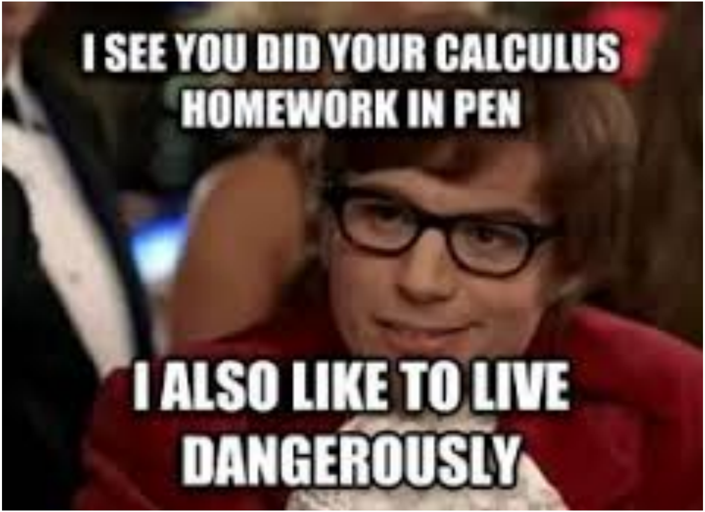
- $f(x) = \frac{2x^2}{x^2-1}$

- $f(x) = \frac{x^2}{x+1}$

- $f(x) = \frac{x^2}{\sqrt{x+1}}$

- $f(x) = \frac{e^x}{x}$

Curve Sketching

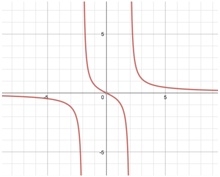


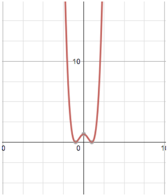
Five in Five!

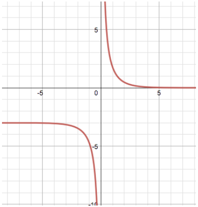
1. Identify where the function $x^3 + x^2 - x$ is increasing or decreasing, and concave up or concave down.

Match the functions below to their graphs:

1. $f(x) = (x^2 - 1)^2$
2. $f(x) = \frac{x}{x^2-4}$
3. $f(x) = \frac{-x}{2x^2+1}$
4. $f(x) = \frac{3}{e^x-1}$







Flex the Mental Muscle!

Use the steps for curve sketching to sketch the curve

$$y = \frac{(x+9)}{\ln x},$$

identifying the domain, any x- and y-intercepts, intervals of increase/decrease, local max/mins, intervals of concave up, concave down, and inflections points.