PART A

Answer each question by putting one of the letters a, b, c, d, e in the appropriate answer box below.

Question #	1	2	3	4	5	6	7	8	9	10	11
Answer											
Value	2	3	3	1	2	2	1	1	2	3	2

[2] 1. Find the equation of the line through (6, 2) and parallel to $-5x + 4y = -38$.

a) $y = -\frac{3}{2}x - \frac{19}{2}$

b) $y = \frac{5}{4}x - \frac{11}{2}$

c) $y = \frac{4}{5}x - \frac{2}{5}$

d) $y = -\frac{5}{4}x + \frac{11}{5}$

e) none of the above

[3] 2. After two years on the job, an engineer's salary was \$50,000. After seven years on the job, her salary was \$66,000. Let y represent her salary after x years on the job. Assuming that the change on her salary over time can be approximated by a straight line, give an equation for this line in the form $y = mx + b$.

a) $y = 16,000x + 50,000$

b) $y = 3200x + 43,600$

c) $y = 16,000x + 18,000$

d) $y = 3200x + 50,000$

e) none of the above

[3] 3. Midtown Delivery Service delivers packages which cost \$1.30 per package to deliver. The fixed cost to run the delivery truck is \$129 per day. If the company charges \$4.30 per package, how many packages must be delivered daily to make a profit of \$54?

a) 43 packages

b) 23 packages

c) 61 packages

d) 99 packages

e) none of the above

[1] 4. Let the demand function of a certain product be $p = 100 - e^x$ where x is the number of units of the product and p is the price in dollars per unit. Then the revenue function is:

a) $R(x) = (100 - e^x)C(x)$

b) $R(x) = 100 - e^x$

c) $R(x) = 100x - e^{x^2}$

d) $R(x) = 100 - e^x - x$

e) $R(x) = 100x - xe^x$

[2] 5. The domain of the function $f(x) = \sqrt{13 - x}$ is:

a) $[0,\ 13]$

b) $(-\infty,\ 13]$

c) $(-\infty,\ 13) \cup (13,\ \infty)$

d) $(-\infty,\ \infty)$

e) none of the above

[2] 6. Use the graph below to find the domain and range of a function $g(x)$

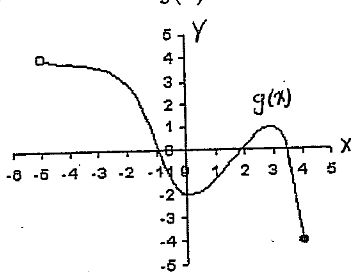
a) Domain $[-4,\ 4)$; Range $(-5,\ 4]$

b) Domain $[-5,\ 4]$; Range $[-4,\ 4]$

c) Domain $(-5,\ 4)$; Range $[-2,\ 4)$

d) Domain $(-5,\ 4]$; Range $[-4,\ 4)$

e) none of the above



[1] 7. Use the graph below to find $\lim_{x \to 0^+} f(x)$

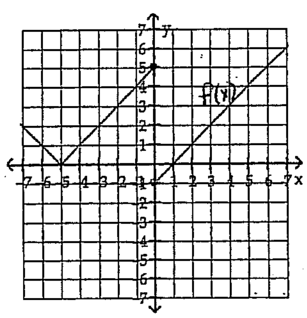
a) -5

b) 5

c) -1

d) 1

e) 0



[1] 8. Use the graph in question 7 above, to find $f(0)$.

a) -5 b) 1 c) 5 d) -1 e) does not exist

[2] 9. Use the properties of limits to help decide whether the limit exists. If the limit exists, find its value.

$$\lim_{x \to -\infty} \frac{4x^3 + 5x}{-6x^4 + 8x^3 + 10}$$

a) $-\infty$ b) $\frac{2}{3}$ c) $-\frac{2}{3}$ d) 0 e) none of the above

[3] 10. Find the value(s) of x for which the tangent line to $y = x^4 - 32x$ is horizontal.

a) 8 b) 0 c) -2, 2 d) does not exist e) 2

[2] 11. If $\log_2 x^2 = 4$ then x has the value:

a) 2 b) $\ln 2$ c) 4 d) $2\sqrt{2}$ e) $\sqrt{\ln 8}$

Part B

Answer the following question in the space provided. Show details of your work for full marks.

[3] 12. (a) How much money does Mr. Taylor need to invest at 8% compounded semiannually, if the investment is to be worth $10,000 in 5 years? (Leave the answer in unsimplified form)

[5] (b) The half-life of radium 226 is approximately 1620 years. A sample of radium 226 weighs 4 grams. Write an exponential formula for the amount of radium 226 remaining in t years. (Leave the answer in logarithmic form.)

13. Find the following limits if they exist. Simplify the answers.

[3] (a) $\lim_{x \to -\frac{1}{2}} \frac{2x^2 + 3x + 1}{2x^2 - x - 1}$

[4] (b) $\lim_{x \to 4} \frac{\sqrt{x} - 2}{x^2 - 16}$

14. Let $f(x)$ be the function

$$f(x) = \begin{cases} 2x^2 - 1, & x < 1 \\ 2, & x = 1 \\ 3x - 2, & x > 1 \end{cases}$$

[3] (a) Find $\lim_{x \to 1^-} f(x)$, $\lim_{x \to 1^+} f(x)$, and $f(1)$.

[2] (b) Is $f(x)$ continuous at $x = 1$? Explain using your answer to part (a).

[6] 15. Use the definition of the derivative of a function to find $f'(x)$ if $f(x) = \sqrt{1 - x}$.

16. Find $f'(x)$ for each of the following functions. (Do not simplify.)

[4] (a) $f(x) = \left(x^2 - 3x\right)\left(x^3 + \pi^2\right)$

[4] (b) $f(x) = \dfrac{x^{\frac{1}{2}} + 10}{3x^2 - 5x}$

[4] (c) $f(x) = 5x^3 + \sqrt[3]{x^2} - x^{-\frac{1}{3}} + \dfrac{6}{x^3}$

MATH 1520: Solutions of Midterm (Oct. 27,2005)

Question 1

Find the equation of the line through $(6, 2)$ and parallel to $-5x + 4y = -38$.

Two parallel lines have the same slope (the same steepness). To find the slope of the line $-5x+4y = -38$, give the Slope-Intercept form of this equation.

$$-5x + 4y = -38$$
$$4y = 5x - 38$$
$$y = \frac{5}{4}x - \frac{38}{4}$$

the slope of the line $-5x + 4y = -38$ is $m = \frac{5}{4}$.

The solution line has a slope $m = \frac{5}{4}$ and goes through the point $(6, 2)$. Now we have 2 alternatives:

- Use the Point-Slope form of the equation with $m = \frac{5}{4}$ and $(x_1, y_1) = (6, 2)$ to find the solution equation

$$(y - y_1) = m(x - x_1)$$
$$(y - 2) = \frac{5}{4}(x - 6)$$
$$(y - 2) = \frac{5}{4}x - \frac{5 \times 6}{4}$$
$$y = \frac{5}{4}x - \frac{30}{4} + 2$$
$$y = \frac{5}{4}x - \frac{30}{4} + \frac{8}{4}$$
$$y = \frac{5}{4}x - \frac{22}{4}$$
$$y = \frac{5}{4}x - \frac{11}{2}$$

  so **b)** is the answer.

- The line proposed in **b)** has a slope equal to $m = \frac{5}{4}$. To know if **b)** is the appropriate answer, the point $(6, 2)$ must satisfy the equation **b)**. So we substitute $(6, 2)$ in $y = \frac{5}{4}x - \frac{11}{2}$

$$2 = \frac{5}{4} \times 6 - \frac{11}{2}$$
$$2 = \frac{15}{2} - \frac{11}{2}$$
$$2 = \frac{4}{2}$$

  $(6, 2)$ is solution of the equation given in **b)**, so **b)** is the answer.

Question 2

Find the equation of the line that goes through points $(2, 50000)$ and $(7, 66000)$.

1. Calculation of the slope by using the definition of the slope for $(x_1, y_1) = (2, 50000)$ and $(x_2, y_2) = (7, 66000)$:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$m = \frac{66000 - 50000}{7 - 2}$$
$$m = \frac{16000}{5}$$
$$m = \frac{16 \times 1000}{5}$$
$$m = 16 \times 200 = 3200$$

   The same result can be obtained by using $(x_1, y_1) = (7, 66000)$ and $(x_2, y_2) = (2, 50000)$.

2. Use the Point-Slope form of the equation of a line with $m = 3200$ and $(x_1, y_1) = (2, 50000)$ to obtain the equation of the solution line.

$$(y - y_1) = m(x - x_1)$$
$$(y - 50000) = 3200(x - 2)$$
$$(y - 50000) = 3200x - 3200 \times 2$$
$$y = 3200x - 6400 + 50000$$
$$y = 3200x + 43600$$

The same result can be obtained by using $(x_1, y_1) = (7, 66000)$.

So **b)** is the answer.

Alternative method: $(2, 50000)$ and $(7, 66000)$ must satisfy the solution equation.

Question 3

Find how many packages must be delivered daily to make a profit of \$54. In other words, find the $x-$value for which the Profit function $P(x)$ is equal to 54: Solve $P(x) = 54$.

Definition of the Profit function $P(x)$

$$P(x) = R(x) - C(x)$$

where x is the number of packages daily delivered, $R(x)$ is the Revenue function and $C(x)$ is the Cost function.

Definition of the Revenue function:

$$R(x) = px$$

where p is the price per package, $p = \$4.3$. So

$$R(x) = 4.3x$$

Definition of the Cost Function $C(x)$: we know the fixed cost $b = \$129$ and the marginal cost (cost per package) $m = \$1.3$.

$$C(x) = mx + b$$
$$C(x) = 1.3x + 129$$

Solve $P(x) = 54$.

$$P(x) = 54$$
$$R(x) - C(x) = 54$$
$$4.3x - 1.3x - 129 = 54$$
$$3x - 129 = 54$$
$$3x = 54 + 129$$
$$3x = 183$$
$$x = \frac{183}{3} = 61$$

So **c)** is the answer.

Question 4

Definition of the Revenue function $R(x)$

$$R(x) = px$$

where p is the price per unit, and x is the number of units. So

$$R(x) = (100 - e^x)x$$
$$R(x) = 100x - xe^x$$

So **e)** is the answer.

Question 5

Domain of $f(x) = \sqrt{13 - x}$?

Square root is only defined for nonnegative values. Then f is only defined if

$$13 - x \geq 0$$
$$13 \geq x$$

then $D_f = (-\infty, 13]$.

So **b)** is the answer.

Question 6

d) is the answer.

Comments: f is not defined at $x = -5$ (light dot). f is defined at $x = 4$, $f(4) = -4$ (heavy dot).

Question 7

c) is the answer.

Comment: look at values of $f(x)$ as x approaches 1 from the right (with values bigger than 1).

Question 8

c) is the answer.

f is defined at $x = 0$, $f(0) = 5$ (heavy dot).

Question 9

$$\lim_{x\to-\infty}\frac{4x^3+5x}{-6x^4+8x^3+10}=\lim_{x\to-\infty}\frac{\frac{4x^3}{x^4}+\frac{5x}{x^4}}{\frac{-6x^4}{x^4}+\frac{8x^3}{x^4}+\frac{10}{x^4}}=\lim_{x\to-\infty}\frac{\frac{4}{x}+\frac{5}{x^2}}{-6+\frac{8}{x}+\frac{10}{x^4}}$$

As $x\to-\infty$,
$\frac{4}{x}\to 0$ and $\frac{5}{x^2}\to 0$, so $\frac{4}{x}+\frac{5}{x^2}\to 0$,
$\frac{8}{x}\to 0$, and $\frac{10}{x^4}\to 0$, so $-6+\frac{8}{x}+\frac{10}{x^4}\to -6$
So

$$\lim_{x\to-\infty}\frac{4x^3+5x}{-6x^4+8x^3+10}=0$$

So **d)** is the answer.

Question 10

An horizontal line has a slope equal to 0. Since the slope of the tangent line to f at x is the first derivative $f'(x)$, we have to solve $f'(x)=0$.

Calculation of f':

$$f'(x)=4x^3-32$$

Solve the equation

$$\begin{aligned}f'(x)&=0\\4x^3-32&=0\\4x^3&=32\\x^3&=8\\x^3&=2^3\\x&=2\end{aligned}$$

So **e)** is the answer.

Question 11

$$\begin{aligned}&\log_2 x^2=4\\&2\log_2 x=4 \qquad (\log_a x^r=r\log_a x)\\&\log_2 x=2\\&x=2^2 \qquad (\log_a x=y\Leftrightarrow x=a^y)\\&x=4\end{aligned}$$

So **c)** is the answer.

Question 12

(a) Using the formula of the compound amount $A(t) = P(1 + \frac{r}{m})^{tm}$, we can calculate P.

$$\begin{aligned} A(t) &= P(1 + \frac{r}{m})^{tm} \\ 10000 &= P(1 + \frac{0.08}{2})^{5\times 2} \\ 10000 &= P(1.04)^{10} \\ \frac{10000}{(1.04)^{10}} &= P \end{aligned}$$

so Mr. Taylor needs to invest $P = \frac{10000}{(1.04)^{10}}$ dollars.

(b) Exponential formula for the amount of radium with respect to time t is

$$y(t) = y_0 e^{kt}$$

where y_0 is the intial amount of radium and k is the decay constant.

Here y_0 is 4 grams. Now, we need to calculate the decay constant k of the radium. The half-time of radium is $t = 1620$ years, so

$$\begin{aligned} \frac{y_0}{2} &= y_0 e^{1620k} \\ \frac{1}{2} &= e^{1620k} \\ \ln\frac{1}{2} &= 1620k \\ \frac{\ln\frac{1}{2}}{1620} &= k \end{aligned}$$

So the amount of radium with respect to time t is given by

$$\begin{aligned} y(t) &= 4e^{\frac{\ln\frac{1}{2}}{1620}t} \\ y(t) &= 4\left(\frac{1}{2}\right)^{\frac{t}{1620}} \end{aligned}$$

Question 13

(a)

$$\lim_{x\to -\frac{1}{2}} \frac{2x^2+3x+1}{2x^2-x-1} = \lim_{x\to -\frac{1}{2}} \frac{(2x+1)(x+1)}{(2x+1)(x-1)} = \lim_{x\to -\frac{1}{2}} \frac{x+1}{x-1} = \frac{-\frac{1}{2}+1}{-\frac{1}{2}-1} = \frac{\frac{1}{2}}{-\frac{3}{2}} = -\frac{1}{3}$$

(b)

$$\begin{aligned} \lim_{x\to 4} \frac{\sqrt{x}-2}{x^2-16} &= \lim_{x\to 4} \frac{(\sqrt{x}-2)(\sqrt{x}+2)}{(x^2-16)(\sqrt{x}+2)} = \lim_{x\to 4} \frac{x-4}{(x^2-16)(\sqrt{x}+2)} = \lim_{x\to 4} \frac{x-4}{(x-4)(x+4)(\sqrt{x}+2)} \\ &= \lim_{x\to 4} \frac{1}{(x+4)(\sqrt{x}+2)} = \frac{1}{(4+4)(\sqrt{4}+2)} = \frac{1}{8\times 4} = \frac{1}{32} \end{aligned}$$

Question 14

(a)

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} 2x^2 - 1 = 1$$

$$\lim_{x\to1^+} f(x) = \lim_{x\to1^+} 3x-2 = 1$$

$$f(1) = 2$$

(b) $\lim_{x\to1^-} f(x) = \lim_{x\to1^+} f(x) = 1$ so $\lim_{x\to1} f(x) = 1$
f is discontinuous at $x=1$ because $\lim_{x\to1} f(x) \neq f(1)$ $(1 \neq 2)$

Question 15

If $f(x) = \sqrt{1-x}$

$$\begin{aligned}
f'(x) &= \lim_{x\to0} \frac{f(x+h)-f(x)}{h} = \lim_{x\to0} \frac{\sqrt{1-(x+h)}-\sqrt{1-x}}{h} \\
&= \lim_{x\to0} \frac{(\sqrt{1-(x+h)}-\sqrt{1-x})(\sqrt{1-(x+h)}+\sqrt{1-x})}{h(\sqrt{1-(x+h)}+\sqrt{1-x})} = \lim_{x\to0} \frac{1-(x+h)-(1-x)}{h(\sqrt{1-(x+h)}+\sqrt{1-x})} \\
&= \lim_{x\to0} \frac{1-x-h-1+x}{h(\sqrt{1-(x+h)}+\sqrt{1-x})} = \lim_{x\to0} \frac{-h}{h(\sqrt{1-(x+h)}+\sqrt{1-x})} \\
&= \lim_{x\to0} \frac{-1}{\sqrt{1-(x+h)}+\sqrt{1-x}} = \frac{-1}{\sqrt{1-x}+\sqrt{1-x}} = \frac{-1}{2\sqrt{1-x}}
\end{aligned}$$

So $f'(x) = \frac{-1}{2\sqrt{1-x}}$

Question 16

(a) The function $f(x) = (x^2-3x)(x^3+\pi^2)$ can be expressed as a product of functions $f(x) = u(x)v(x)$ where $u(x) = (x^2-3x)$ and $v(x) = (x^3+\pi^2)$. Using the Product Rule, we calculate f':

$$f'(x) = u(x)v'(x) + v(x)u'(x) = (x^2-3x)(3x^2) + (x^3+\pi^2)(2x-3)$$

(b) The function $f(x) = \frac{x^{1/2}+10}{3x^2-5x}$ can be expressed as a quotient of functions $f(x) = \frac{u(x)}{v(x)}$ where $u(x) = x^{1/2}+10$ and $v(x) = 3x^2-5x$. Using the Quotient Rule, we calculate f':

$$f'(x) = \frac{v(x)u'(x)-u(x)v'(x)}{v(x)^2} = \frac{(3x^2-5x)(1/2x^{-1/2})-(x^{1/2}+10)(6x-5)}{(3x^2-5x)^2}$$

(c) The function $f(x) = 5x^3 + x^{\frac{2}{3}} - x^{-\frac{1}{3}} + 6x^{-3}$ can be expressed as a sum of power functions. Using the Power Rule, we calculate f':

$$f'(x) = 5\times 3x^{3-1} + \frac{2}{3}x^{\frac{2}{3}-1} - 1\times(\frac{-1}{3})x^{-\frac{1}{3}-1} + 6\times(-3)x^{-3-1}$$
$$f'(x) = 15x^2 + \frac{2}{3}x^{-\frac{1}{3}} + \frac{1}{3}x^{-\frac{4}{3}} - 18x^{-4}$$