UNIVERSITY OF MANITOBA

DATE: February 26, 2009 | MIDTERM EXAM

COURSE: MATH 1520 | TIME: 1 hours

EXAMINATION: Intro. Calc. for Mgmt & Soc. Sci. | EXAMINER: Various

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PART A: Multiple Choice Questions

Answer each question by putting one of the letters A, B, C, D, E in the appropriate answer box below (2 point each).

Question #	1	2	3	4	5	6	7	8
Answer	B	A	D	~~C~~ D	~~D~~ C	B	C	A

[2] 1. What is the domain of the function $f(x) = \dfrac{x-2}{\sqrt{3x-1}}$?

A. $(-\infty, \frac{1}{3}) \cup (\frac{1}{3}, \infty)$;

B. $(\frac{1}{3}, \infty)$;

C. $[\frac{1}{3}, \infty)$;

D. $(-\infty, \frac{1}{3})$;

E. None of the above.

[2] 2. Let $f(x) = x^2 + 3x - 3$. What is the equation of the tangent line to the curve $y = f(x)$ at the point $(1, 1)$ given that $f'(x) = 2x + 3$?

A. $y = 5x - 4$;

B. $y = 5x$;

C. $y = 2x - 1$;

D. $x = 5y - 4$;

E. None of the above.

Questions 3, 4, and 5 refer to the following scenario:

The Naive Company produces and sells bottled spring water. The marginal cost of producing a bottle of water is \$2, and it costs \$50 to produce 20 bottles of water. Each bottle of water is sold for \$3.

[2] 3. What is a linear cost function for producing x bottles of water?

A. $C(x) = 3x - 10$;

B. $C(x) = x + 30$;

C. $C(x) = \dfrac{1}{2}x + 40$;

D. $C(x) = 2x + 10$;

E. None of the above.

[2] 8. How long will it take for an investment of \$100 to grow to \$300 in an account making interest at an annual rate of 12% compounded quarterly?

A. $t = \dfrac{\ln 3}{4\ln(1.03)}$;

B. $t = \dfrac{4\ln(1.03)}{\ln 3}$;

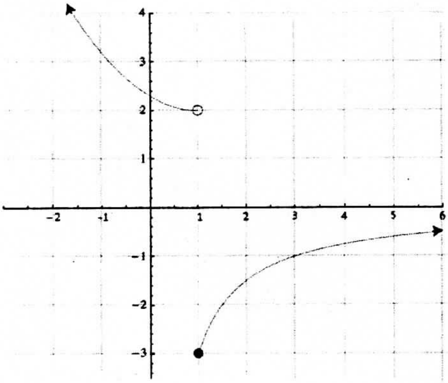
C. $t = \dfrac{3\ln(1.04)}{\ln 3}$;

D. $t = \dfrac{\ln 3}{3\ln(1.12)}$

E. None of the above.

PART B: Short Answer Questions

9. Write your final answers for this question on the lines provided below. Only your final answers will be marked!



The above is the graph of the function $f(x)$. The graph goes up without bound as it approaches the line $x = -2$, and it approaches the x-axis to the right.

Use the graph to find the value of the quantities below. If a limit does not exist, indicate whether it tends to $+\infty$, $-\infty$, or neither.

[1] (a) $\lim_{x\to 1^-} f(x) = \underline{\ 2\ }$.

[1] (b) $\lim_{x\to 1^+} f(x) = \underline{\ -3\ }$.

[1] (c) $\lim_{x\to 1} f(x) = \underline{\text{DNE}}$.

[1] (d) $\lim_{x\to \infty} f(x) = \underline{\ 0\ }$.

[1] (e) $\lim_{x\to -2^+} f(x) = \underline{\ \infty\ }$.

PART C: Long Answer Questions

Please show your work clearly.

If necessary, you may continue your work on the reverse sides of the pages of this exam or on the blank pages at the end—but please indicate clearly where your work may be found.

10. The half-life of a radioactive substance is 3 years.

[4] (a) Find the decay constant, that is, the number r such that the function e^{rt} describes the decay of the radioactive substance.

Solution: The decay of an initial amount of y_0 radioactive material after time t is given by $y(t) = y_0e^{rt}$. We were given that its half-time is 3 years, and so after 3 years the substance decays to half of its original size; in other words, $y(3) = \frac{1}{2}y_0$. Thus,

$$\begin{aligned} y_0e^{3r} &= \frac{1}{2}y_0 && 1 \\ e^{3r} &= \frac{1}{2} && 1 \\ \ln e^{3r} &= \ln\frac{1}{2} \\ 3r &= -\ln 2 && 1 \rightarrow \frac{\ln(1/2)}{3} \\ r &= -\frac{\ln 2}{3}. && 1 \end{aligned}$$

[4] (b) If a sample of this substance had an initial mass of 80g, find the **exact value** of the mass remaining after 9 years.

Solution: We found r in part (a). We are given that the initial amount of the substance is $y_0 = 80$, and the question is asking for $y(9)$. 1

$$y(9) = 80e^{(-\frac{\ln 2}{3})9} = 80e^{-3\ln 2} = 80e^{-\ln 2^3} = 80e^{\ln\frac{1}{8}} = 80\cdot\frac{1}{8} = 80.$$

1 1 1

Thus, after 9 years, 10g of radioactive material will be left.

$(e^{\ln 2})^{-3}$

→ OR halve 80 three times!

11. Let $f(x) = \dfrac{1}{x^2}$.

[7] (a) Find $f'(x)$ for $x \neq 0$ **using the definition of the derivative.**

No lim -2

Solution:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h}$$

$$= \lim_{h\to 0} \frac{\frac{x^2-(x+h)^2}{x^2(x+h)^2}}{h} = \lim_{h\to 0} \frac{x^2-(x+h)^2}{hx^2(x+h)^2}$$

$$= \lim_{h\to 0} \frac{x^2-(x^2+2xh+h^2)}{hx^2(x+h)^2}$$

$$= \lim_{h\to 0} \frac{-2xh-h^2}{hx^2(x+h)^2} = \lim_{h\to 0} \frac{-2x-h}{x^2(x+h)^2}$$

As $h \to 0$, the numerator approaches $2x$ and the denominator approaches x^4. Since $x \neq 0$, $x^4 \neq 0$, and so the limit in question is equal to the quotient of the limits. Therefore, $f'(x) = -\frac{2}{x^3}$. Not req'd.

OR

$$f'(x) = \lim_{b\to x} \frac{f(b)-f(x)}{b-x}$$

$$= \lim_{b\to x} \frac{\frac{1}{b^2}-\frac{1}{x^2}}{b-x}$$

$$= \lim_{b\to x} \frac{x^2-b^2}{b^2x^2(b-x)}$$

$$= \lim_{b\to x} \frac{(x+b)(x-b)}{b^2x^2(b-x)}$$

$$= \lim_{b\to x} \frac{-(x+b)}{b^2x^2}$$

$$= \frac{-2x}{x^4}$$

$$= -\frac{2}{x^3}$$

[1] (b) Is $f(x)$ differentiable at $x = 0$? (No marks for an unexplained answer!)

Solution: Since $f(x)$ is not defined at $x = 0$, it is not differentiable there either.

[2] (c) Find the equation of the tangent line to the graph of $y = f(x)$ at $(\frac{1}{2}, f(\frac{1}{2}))$.

Solution: We have $f(\frac{1}{2}) = 4$ and $f'(\frac{1}{2}) = -\frac{2}{(\frac{1}{2})^3} = -16$. Thus, the equation of the tangent line is:

$$y - 4 = -16(x - \frac{1}{2})$$

$$y - 4 = -16x + 8$$

$$y = -16x + 12.$$

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12. Find the following limits. If they do not exist, state whether they are ∞ or $-\infty$ if possible.

[6] (a) $\lim_{x\to\infty} \sqrt{2x^2-3x} - \sqrt{2x^2+2}$

Solution: By multiplying both numerator and denominator of the expression by $\sqrt{2x^2-3x}+\sqrt{2x^2+2}$, and obtain that the limit in question is equal to

$$\lim_{x\to\infty} \frac{(\sqrt{2x^2-3x}-\sqrt{2x^2+2})(\sqrt{2x^2-3x}+\sqrt{2x^2+2})}{\sqrt{2x^2-3x}+\sqrt{2x^2+2}} \quad 1$$
$$= \lim_{x\to\infty} \frac{(2x^2-3x)-(2x^2+2)}{\sqrt{2x^2-3x}+\sqrt{2x^2+2}} \quad 1$$
$$= \lim_{x\to\infty} \frac{-3x-2}{\sqrt{2x^2-3x}+\sqrt{2x^2+2}} \quad 1$$
$$= \lim_{x\to\infty} \frac{-3-\frac{2}{x}}{\sqrt{2-\frac{3}{x}}+\sqrt{2+\frac{2}{x^2}}}. \quad 1$$

The denominator approaches to $\sqrt{2}+\sqrt{2} = 2\sqrt{2} \neq 0$ as $x \to \infty$, 1 and it is equal to the quotient of limits: $\frac{-3}{2\sqrt{2}}$. 1

[3] (b) $\lim_{x\to 2} e^{\frac{1}{x^2-4}}$ $e^{\left(\frac{1}{x^2-4}\right)}$ NOT $e^{\frac{1}{x^2-4}}$

Solution: When $x \to 2^+$, we have $x^2-4 \to 0^+$, and so $\lim_{x\to 2^+} \frac{1}{x^2-4} = \infty$. 1

Thus, $\lim_{x\to 2^+} e^{\frac{1}{x^2-4}} = \infty$.

On the other hand, when $x \to 2^-$, we have $x^2-4 \to 0^-$, and so $\lim_{x\to 2^-} \frac{1}{x^2-4} = -\infty$. Consequently, $\lim_{x\to 2^-} e^{\frac{1}{x^2-4}} = 0$. 1

Since $\lim_{x\to 2^-} e^{\frac{1}{x^2-4}} \neq \lim_{x\to 2^+} e^{\frac{1}{x^2-4}}$, the two-sided limit $\lim_{x\to 2} e^{\frac{1}{x^2-4}}$ does not exist. 1

[3] (c) $\lim_{x\to\sqrt{2}} \frac{(x-\sqrt{2})(x^2+3x+2)}{x^2-2}$

Solution:

$$\lim_{x\to\sqrt{2}} \frac{(x-\sqrt{2})(x^2+3x+2)}{x^2-2} = \lim_{x\to\sqrt{2}} \frac{(x-\sqrt{2})(x^2+3x+2)}{(x-\sqrt{2})(x+\sqrt{2})} \quad 1$$
$$= \lim_{x\to\sqrt{2}} \frac{x^2+3x+2}{x+\sqrt{2}}. \quad 1$$

Since the denominator approaches to $2\sqrt{2} \neq 0$ as $x \to \sqrt{2}$, the limit is equal to the quotient of the limits: $\frac{2+3\sqrt{2}+2}{2\sqrt{2}} = \frac{3\sqrt{2}+4}{2\sqrt{2}} = \frac{3}{2}+\sqrt{2}$. 1

13. Let $f(x) = \begin{cases} 3x^2 + Kx & \text{if } x < 1 \\ 5 & \text{if } x = 1 \\ K^2x + \dfrac{2x-2}{x^2-1} & \text{if } x > 1. \end{cases}$

[5] (a) Find both values of K such that $\lim_{x \to 1} f(x)$ exists.

Solution: We need to have $\lim_{x\to 1^-} f(x) = \lim_{x \to 1+} f(x)$. When $x \to 1^-$, we have $x < 1$, and so

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} 3x^2 + Kx = 3 + K, \quad (1)$$

because $3x^2 + Kx$ is a polynomial. When $x \to 1^+$, then $1 < x$. Thus,

$$\lim_{x\to 1+} f(x) = \lim_{x\to 1+} K^2x + \frac{2x-2}{x^2-1} = \lim_{x\to 1+} K^2x + \frac{2(x-1)}{(x-1)(x+1)} \quad (1)$$
$$= \lim_{x\to 1+} K^2x + \frac{2}{x+1} = K^2 + \frac{2}{2} = K^2 + 1. \quad (1)$$

because $\lim_{x\to 1+} x + 1 \neq 0$. The two one-sided limits are equal precisely when $3 + K = K^2 + 1$, that is,

$$K^2 - K - 2 = 0. \quad (1)$$

Hence, $K = 2$ or $K = -1$. (1)

[4] (b) Find a value of K (if it exists) such that $f(x)$ is continuous at $x = 1$.

Solution: In order for $f(x)$ to be continuous at the point $x = 1$, we need $f(1) = \lim_{x\to 1} f(x)$, and in particular, the limit $\lim_{x\to 1} f(x)$ must exist and be finite. By part (a), this already implies that $K = 2$ or $K = -1$. (1)

For $K = -1$, one has $\lim_{x\to 1} f(x) = K^2 + 1 = 2 \neq 5 = f(1)$. Thus, for $K = -1$, $f(x)$ is not continuous at $x = 1$. (1) +1?

For $K = 2$, one has $\lim_{x\to 1} f(x) = K^2+1 = 5 = f(1)$. Therefore, for $K = 2$, $f(x)$ is continuous at $x = 1$. (1)

OR Set $\lim_{x\to 1^+} f(x) = \lim_{x \to 1^-} f(x) = f(1)$ (1)

$\underbrace{K^2+1 = 3+K}_{(1)} = 5 \Rightarrow K = 2$ (1)

(1)

END OF EXAM