CHAPTER 7: INTEGRATION

Section 7.1: Antiderivatives

Say we know the velocity of a particle, but we want to know its position at a given time, or, we know the rate at which a population is growing, but we want to know the size at a given time. In other words, can we work backwards from the derivative, to the original fcn?

A fcn F is called an "antiderivative" of f on an interval if $f(x) = F'(x)$ (ie, $f(x)$ is the derivative of $F(x)$).

So if F & G are any 2 antiderivatives of f, then $F'(x) = f(x) = G'(x)$ so $G(x) - F(x) = C$.

Thm: If F is an antiderivative of f on an interval I, then the most general antideriv. of f on I is

$F(x) + C$ ← an arbitrary constant

By assigning specific values to C, we obtain a family of fcn's whose graphs are vertical translates of one another.

ex/ Draw members of the family of antiderivatives of $f(x) = x^2$.

Table of Antiderivatives:

Fcn'	Anti-deriv.	Fcn'	Anti deriv.
$x^n \ (n \neq -1)$	$\frac{x^{n+1}}{n+1}$	a^{kx}	$\frac{a^{kx}}{k(\ln a)}$
$1/x$	$\ln\lvert x\rvert$	c	cx
e^{kx}	$\frac{e^{kx}}{k}$		

Antideriv. Rules:

1. $cf(x)$ has antideriv. $cF(x)$

2. $f(x) \pm g(x)$ has antideriv. $F(x) \pm G(x)$

ex// Find all fcn's g such that

$$g'(x) = \frac{2x^5 - \sqrt{x}}{x}$$

In applications of calculus, it is very common to have a situation where it is required to find a fcn', given knowledge about its derivative. An eqn' that involves the derivatives of a fcn' is called a "differential equation".

In some cases, there may be some extra conditions given that will determine the constant(s) [C] & therefore uniquely specify the solution.

ex// Find f if $f'(x) = e^x + 20(1+x^2)$ & $f(0) = -2$.

Notice: when we started with f'', we needed 2 pieces of info. because we ended up with 2 unknown constants. In fact, when finding f from $f^{(n)}$ (n^{th} derivative), we will end up with n unknown constants & therefore we will need n conditions (of the type $f(\sim) = \sim$ or $f'(\sim) = \sim$) to find a particular soln'.

Rectilinear Motion:

Recall that if an object has position fcn' $s = f(t)$, velocity is $v(t) = s'(t)$. So position is the antiderivative of velocity. Likewise, acceleration $a(t) = v'(t) = s''(t)$ so velocity is the antiderivative of acceleration!

Now we can find $v(t)$ or $s(t)$ from $a(t)$...

Section 7.3: Area & the Definite Integral

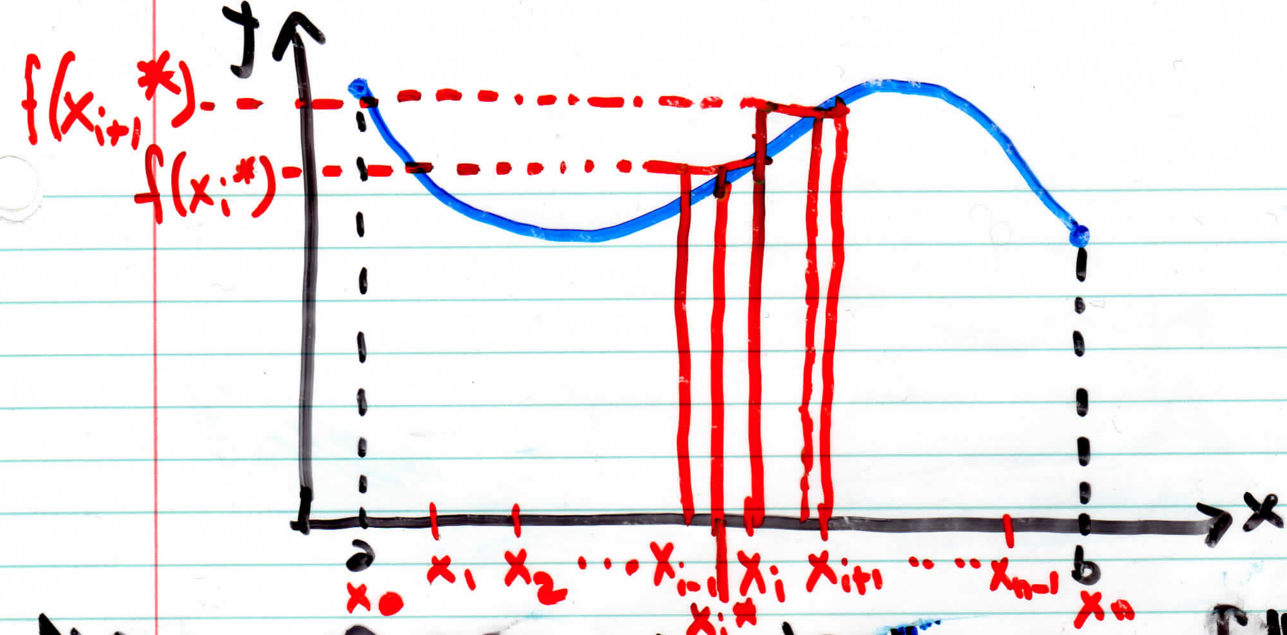
In this section we discover that in trying to find the area under a curve, we end up with a special type of limit.

? What is the *Area Problem* ?

We want to find the area of the region S that lies under a curve $y=f(x)$ $[f(x) \geq 0]$ from $x=a$ to $x=b$, & above the x-axis.

We can make our definition even more general by not specifying where in the interval we choose to draw the rectangle from (make it random) & by letting the area be bounded by a general function $y=f(x)$ & the lines $x=a$ to $x=b$

We start by again dividing $[a,b]$ into n strips of equal width. The width of the interval is $b-a$, so each strip has width $\Delta x = \frac{b-a}{n}$.



Now, we approximate the area of the i^{th} strip S_i with a rectange with width Δx & height $f(x_i^*)$, where $f(x_i^*)$ is the value of f at any number in the i^{th} subinterval $[x_{i-1}, x_i]$. We call these numbers $x_1^*, x_2^*, \ldots, x_n^*$, "sample point". So...

Now, we can use our "integral" notation & our knowledge of anti-derivatives to make this formula more practical. The "definite integral" of f from x=a to x=b is:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

So far, we have restricted ourselves to the case where $f(x) \geq 0$. We can also define the integral if this is not so.

ex// Evaluate the following integrals by interpreting them in terms of area.

a) $\int_0^1 \sqrt{1-x^2}\, dx$

b) $\int_0^3 (x-1)\,dx$

Properties of the Definite Integral:

1 $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

2 $\int_a^a f(x)\,dx = 0$

③ $\int_a^b c\,dx = c(b-a)$

④ $\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

⑤ $\int_a^b c f(x)\,dx = c\int_a^b f(x)\,dx$ (c - constant)

ex/ Use the properties above to evaluate

$\int_0^1 (4+3x^2)\,dx$

⑥ $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

ex// If we know $\int_0^{10} f(x)dx = 17$ & $\int_0^8 f(x)dx = 12$,

find $\int_8^{10} f(x)dx$.

Comparison Properties:

7. If $f(x) \geq 0$ for $a \leq x \leq b$, then $\int_a^b f(x)dx \geq 0$

($\&$, if $f(x) \leq 0$ on $a \leq x \leq b$, $\int_a^b F(x)dx \leq 0$).

8. If $f(x) \geq g(x)$ for $a \leq x \leq b$, then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$

9. If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then

$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$$

ex// Use Property 9 to estimate $\int_0^1 e^{-x^2} dx$

Section 7.4: The Fundamental Theorem of Calculus

We start by looking at a fcn' defined by $g(x) = \int_a^x f(t)dt$, where f is cts on $[a,b]$ & x varies between a & b. g depends only on x. If x is fixed, $\int_a^x f(t)dt$ is a number, but if we let x vary, $\int_a^x f(t)dt$ also varies, & defines a fcn' of x, denoted by $g(x)$.

Fundamental Thm' Part 1:

If f is cts on [a,b], then the fcn' defined by $g(x)=\int_a^x f(t)\,dt$, $a\le x\le b$ is cts. on [a,b] & differentiable on (a,b) & $g'(x)=f(x)$

ex// Find the derivative of $g(x)=\int_0^x \sqrt{1+t^2}\,dt$

ex// Find $\left(\int_x^2 \sqrt[3]{t}\cdot e^t\,dt\right)'$

ex// Find $\left(\int_2^{x^4} \frac{t^2+1}{7-t}\,dt\right)'$

ex// Find $\left(\int_{e^x}^0 \sqrt{1+r^3}\,dr\right)'$

FUNDAMENTAL THM' PART 2:

If f is cts. on $[a,b]$, then $\int_a^b f(x)\,dx = F(b) - F(a)$ where F is <u>any</u> antiderivative of f.

ex// Evaluate $\int_1^3 e^x\,dx$

ex// Evaluate

a) $\int_{-2}^{8} \sqrt[3]{x}\,dx$

Area: If we are using the definate integral to find area, we want the answer to be positive

Steps for finding Area

1. Sketch (if you can)
2. Find the x-intercepts in [a, b], these divide our total region into subregions that are either positive or negative
3. use seperate integrals to find the area of f(x) above the x-axis & below it (which will be separated by the x-intercepts found in 2)
4. Take $|\int f(x)dx|$ (absolute value) for those regions where f(x) was below the x-axis, then sum up all the integrals to get the total area.

ex// Find the area of the shaded region