Chapter 6: The Definite Integral

Section 6.1: Sigma Notation

"Sigma notation" is used to represent a sum of terms, all of which are similar in form.

ex// Write the following in sigma notation.

a) $\frac{1}{1+2^2} + \frac{2}{1+3^2} + \frac{3}{1+4^2} + \frac{4}{1+5^2} + \frac{5}{1+6^2} + \frac{6}{1+7^2}$

b) $\frac{1}{2\cdot 3} + \frac{4}{3\cdot 4} + \frac{9}{4\cdot 5} + \frac{16}{5\cdot 6} + \cdots + \frac{169}{14\cdot 15}$

ex// Write out the first 5 terms of $\sum_{n=1}^{100} \frac{2^n}{n!}$.

We don't always need to begin the sum from 1, & the representations (in sigma notation) are not unique.

ex// Find another representation for

a) $\displaystyle\sum_{n=1}^{10} \frac{(n+4)^2}{e^{n+4}}$ b) $\displaystyle\sum_{j=-3}^{102} \frac{j^2+2j+5}{\sin(j+5)}$

<u>Thm</u>: If $f(i)$ & $g(i)$ are fcn's of i, & m & n are positive integers such that $n>m$, then $\displaystyle\sum_{i=m}^{n}[f(i)+g(i)] = \sum_{i=m}^{n} f(i) + \sum_{i=m}^{n} g(i)$, and

$$\sum_{i=m}^{n} cf(i) = c\sum_{i=m}^{n} f(i) \quad (c \text{ constant})$$

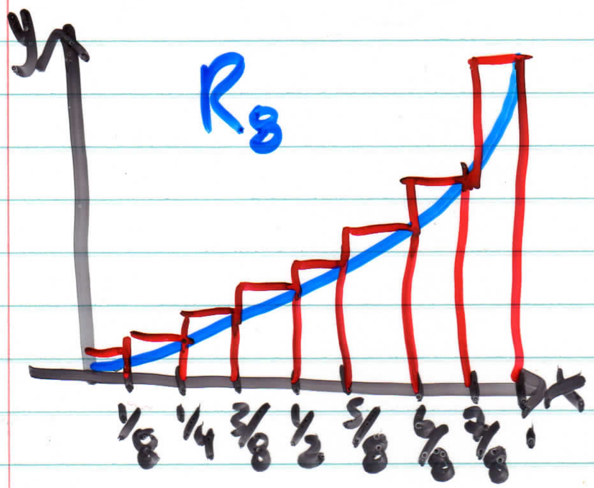
Section 6.3: The Definite Integral

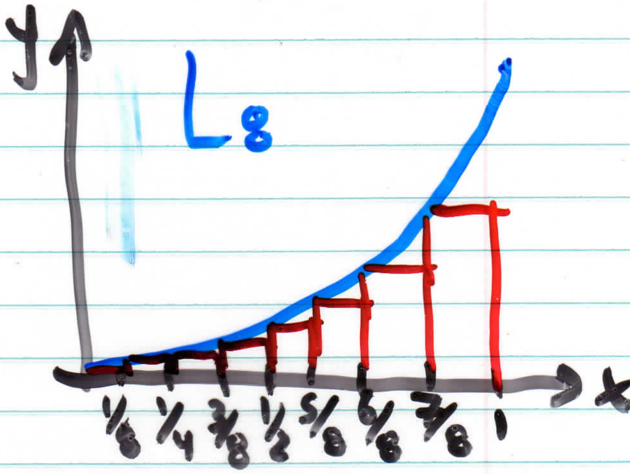
In this section we discover that in trying to find the area under a curve, we end up with a special type of limit.

? What is the *Area Problem* ?

We want to find the area of the region S that lies under a curve $y=f(x)$ $[f(x) \geq 0]$ from $x=a$ to $x=b$.

? How can we make a better estimate ?



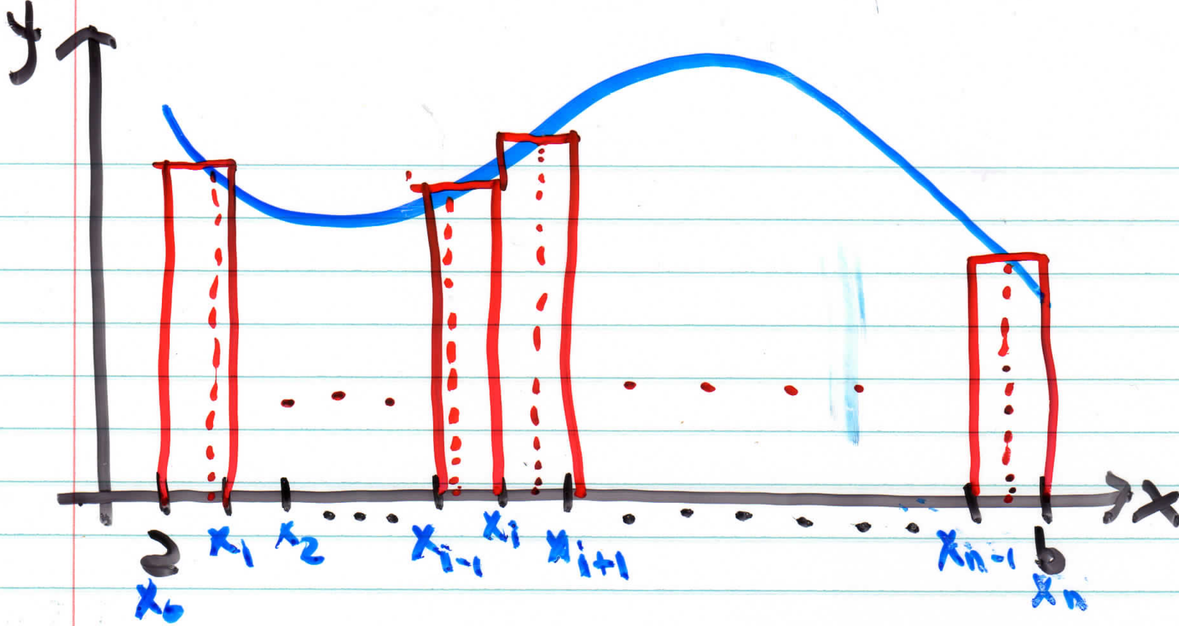


If we used 1000 rectangles, our upper estimate would be 0.3338 & our lower would be 0.3328:

If we took an ∞ number of rectangles, we would get an exact area! Can we prove it?

R_n is the sum of the areas of the n rectangles. Each rectangle has width $\frac{1}{n}$ & height which is the value of $f(x)=x^2$ at the endpoints $\frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \ldots, \frac{n}{n} \Rightarrow$

We can make our defn' even more geneal by not specifying where in the interval we choose to draw the rectangle from (random), & by letting the area be bounded by a geneal fcn' $y=f(x)$ & the lines $x=a$ & $x=b$. We start again by dividing into n strips of equal width ⇒



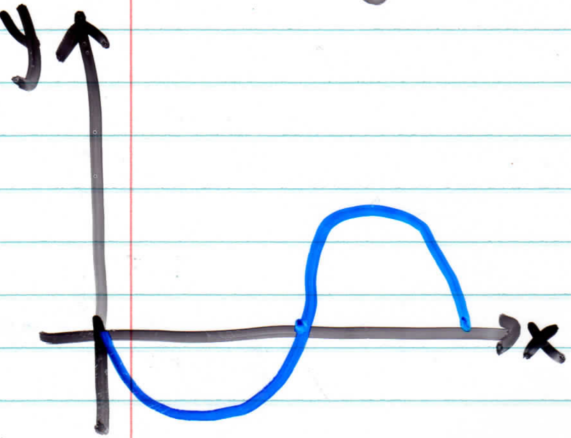
Now, we approximate the area of the i^{th} strip (S_i) with a rectangle with width Δx & height ($f(x_i^*)$) where $f(x_i^*)$ is the value of f at any number in the i^{th} subinterval $[x_{i-1}, x_i]$. We call these numbers $x_1^*, x_2^*, \ldots, x_n^*$ "sample points".

Our general expression becomes:

Defn': If f is a cts fcn' defined for $a \le x \le b$, we divide the interval $[a,b]$ into n subintervals of equal width $\Delta x = \frac{b-a}{n}$. Let $x_0 = a$ & $x_n = b$ be the endpoints of these subintervals, & let $x_1^*, x_2^*, \ldots, x_n^*$ be any sample points in these subintervals, so x_i^* lies in the i^{th} subinterval $[x_{i-1}, x_i]$. Then the "definite integral" of $f(x)$ from a to b is:

$$\int_a^b f(x)\,dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

So far, we have restricted ourselves to the case where $f(x) \geq 0$. We can also define the integral if this is not so:



ex// Evaluate the following integrals by interpreting each in terms of area:

a) $\int_0^1 \sqrt{1-x^2}\,dx$ b) $\int_0^3 (x-1)\,dx$

Section 6.4: The First Fundamental Theorem of Integral Calculus

If f is cts on [a,b], then $\int_a^b f(x)dx = F(b) - F(a)$ where F is any antiderivative of f(x).

ex// Evaluate:

a) $\int_1^3 e^x dx$ b) $\int_0^3 (x-1)dx$ c) $\int_{-\pi/2}^{\pi/2} \sin x\, dx$

Properties of the Definite Integral

1. $$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

2. $$\int_a^a f(x)\,dx = 0$$

3. $$\int_a^b c\,dx = c(b-a)$$

4. $$\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

5. $$\int_a^b c f(x)\,dx = c\int_a^b f(x)\,dx \quad (c \text{ constant})$$

6. $$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Comparison Properties:

(7) If $f(x) \geq 0$ on $a \leq x \leq b \Rightarrow \int_a^b f(x)\,dx \geq 0$

" $f(x) \leq 0$ " " $\Rightarrow$ " ≤ 0

(8) If $f(x) \geq g(x)$ on $a \leq x \leq b$

$\Rightarrow \int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$

(9) If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$$

ex, Estimate $\int_0^1 e^{-x^2}\,dx$.

Velocity & Speed:

Suppose $v(t) = 3t^2 - 6t - 105$, $t \geq 0$. We can calculate the definite integral of $v(t)$ between any two times, say $t = 0$ & $t = 12$:

This is the sum of products of velocities $v(t)$ multiplied by small increments of time, dt:

In general, for $v(t)$ being velocity along the x-axis:

$\int_a^b v(t)\,dt$ is displacement at time $t=b$ relative to displacement at $t=a$.

If we integrate $|v(t)| =$ speed:

In general: $\int_a^b |v(t)|\,dt$ is the distance travelled b/w times $t=a$ & $t=b$.

Section 6.7: Change of Variable in the Definite Integral

We can still use the FTC of Integral Calc. if we use substitution to solve the integral, we can do this by changing the *limits of integration* as well.

Thm: suppose $f(x)$ is cts. on $a \le x \le b$, & we set $x = g(u)$, with $a = g(\alpha)$ & $b = g(\beta)$, then

$$\int_a^b f(x)\,dx = \int_\alpha^\beta f(g(u)) \cdot g'(u)\,du,$$

if $g'(u)$ is cts. on $\alpha \le t \le \beta$, & if when u is b/w α & β, $g(u)$ is b/w a & b.

ex// Solve the following:

a) $\displaystyle\int_{-2}^{1} \frac{x+1}{(x^2+2x+2)^{1/3}}\,dx$

b) $\displaystyle\int_{\pi/4}^{\pi/2} \frac{\sin x}{(1+\cos x)^4}\,dx$

c) $\displaystyle\int_{-4}^{-1} \frac{x}{\sqrt{x+5}}\,dx$