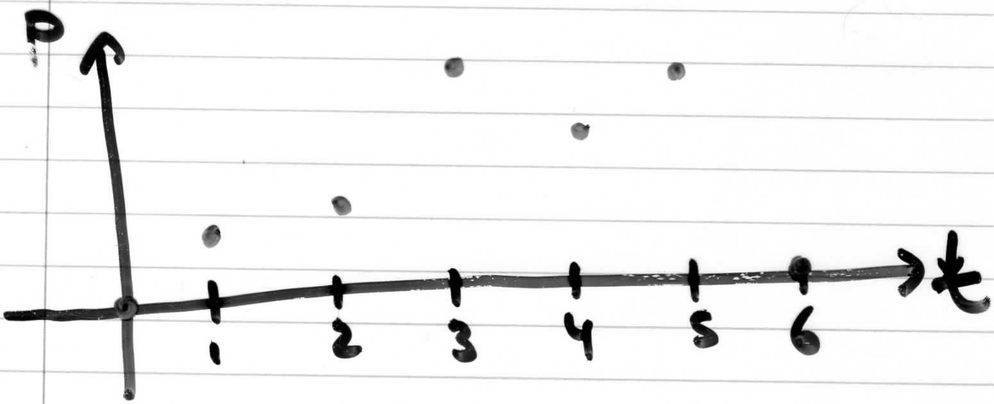
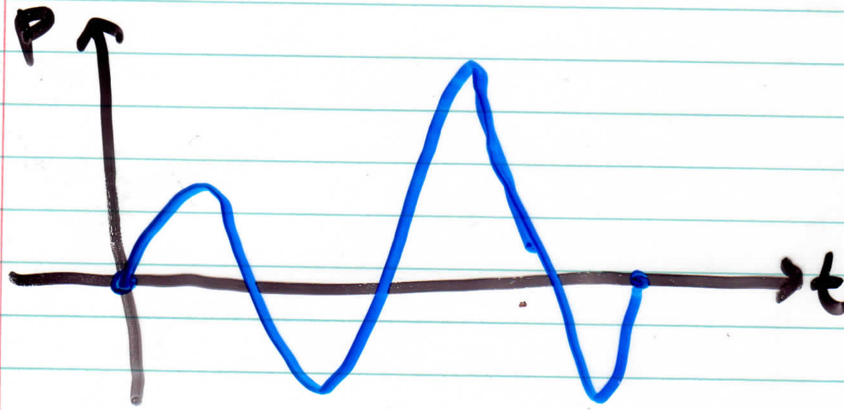
Chapter 3: DIFFERENTIATION

You have designed & built a robotic arm that is currently making repairs to a satellite orbiting the earth.
You have a few snapshots of the arm's position at various times, & are curious as to how fast it is moving.
You plot its position at 1 min intervals



Section 3.1 : The Derivative

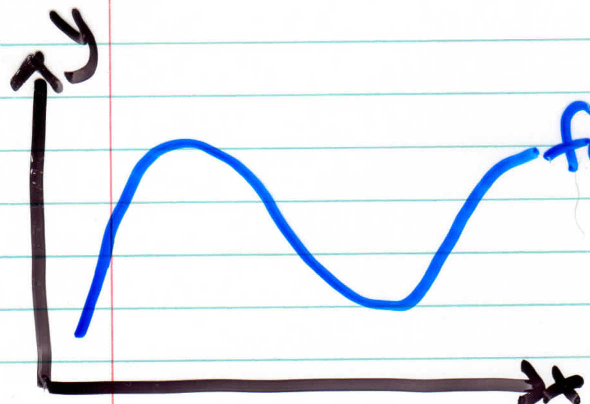
Calculus can help us to trace & analyze the motion of particles & objects moving under the influence of different forces.

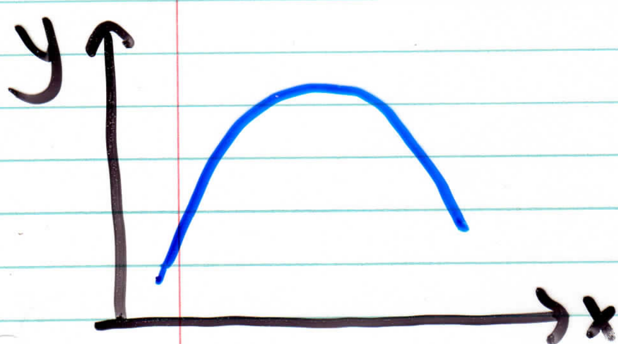


To analyze speed, we can look at p vs t :

ex// find average velocity from $t = 1\,s$ to $t = 2\,s$ of a particle with eqn' of motion $f(t) = t^2 + 2t - 2$.
What is its instantaneous velocity at time $t=1$?

A "tangent" to a curve at a point is a line that touches the curve at that point.





So far, we defined the derivative of a fcn' f at a *fixed* number, x=a:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

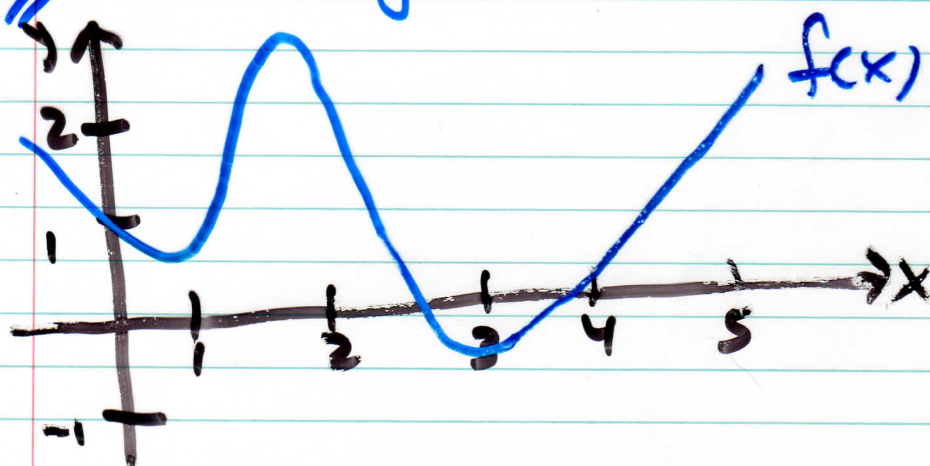
If we replace a in the above eqn' by a variable, x, then we can vary a:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Now, the derivative f'(x) is also a fcn' of x

Use the graph of f(x) to sketch f'(x):

ex/



Some other notations we use for the derivative of $y = f(x)$:

Defn': A fcn' f is "differentiable" at $x=a$ if $f'(a)$ exists. It is differentiable on an interval (a,b) [or (a,∞), $(-\infty,a)$, $(-\infty,\infty)$] if it is differentiable at every number in that interval.

In general, if the graph of a fcn' $f(x)$ has a corner or kink then it is NOT diff'ble at that point.

ex/ In our last example, where is $f(x)$ differentiable?

Section 3.2: Differentiation Rules

We can find derivatives in a faster way than using the limit defn' of the derivative with the following formulas:

① $\frac{d}{dx}(c) = 0$

② $\frac{d}{dx}(x^n) = nx^{n-1}$

ex// find the derivatives of:

a) $f(x) = x^6$

b) $f(x) = x^{100}$

c) $f(x) = \frac{1}{x}$

d) $f(x) = \sqrt{x}$

e) $f(x) = \sqrt[3]{x^4}$

Also, $\frac{d}{dx}[cf(x)] = c\frac{d}{dx}f(x) = cf'(x)$ and

$\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x) = f'(x) \pm g'(x)$.

Section 3.3: Differentiability & Continuity

ex/ Use the graph of $|x|$ to discuss continuity & differentiability.

<u>Theorem</u>: If f is differentiable at a, then f is also continuous at a.

We can use tangent lines to find the angle between 2 intersecting curves using the formula $\theta = \tan^{-1}\left|\dfrac{m_1 - m_2}{1 + m_1 m_2}\right|$

Section 3.4: The Product & Quotient Rules

Suppose we have 2 functions:

$f(x) = x$ & $g(x) = x^2$.

We can form their product: $f \cdot g = x \cdot x^2 = x^3$

& their quotient: $\frac{f}{g} = \frac{x}{x^2} = \frac{1}{x} = x^{-1}$

Section 3.5: Higher-Order Derivatives

We can take a derivative of a derivative:

& our notations are:

ex// find f''' if a) $f(x) = x^4 + 9x^3 - 2$

b) $f(x) = xe^x$

ex// find $f^{(27)}$ if a) $f(x) = \cos x$

b) $f(x) = x^9 + e^x$

Section 3.6: Velocity, Speed, Acceleration

In general, we can interpret the second derivative as a rate of of change OF a rate of change.
So if the 1st deriv. represents velocity, then the 2nd deriv. represents acceleration.

ex// The position of a particle is given by the eqn' $s = f(t) = t^3 - 6t^2 + 9t$
(t - seconds, s - meters).
Find the acceleration at time t.
What is the acceleration at $t = 4$ seconds?

Section 3.7: Chain Rule & Extended Power Rule

So far, we can calculate some derivatives, but now we will calculate derivatives of fcn's inside other fcn's, called "composite fcn's", $f \circ g$ or $f(g(x))$.

ex// find $f \circ g$, $g \circ f$, $f \circ f$ if $f(x) = x^2 + 1$ & $g(x) = \sqrt{x}$.

So, if f & g are both diff'ble & $F = f \circ g$ is the composite fcn', then F is diff'ble &

$$F'(x) = f'(g(x)) \cdot g'(x)$$

All we need to do is figure out which is the inside fcn' & which is the outside fcn'.

Section 3.8: Implicit Differentiation

So far, all of the fcn's we have been working with have been of the type $y=f(x)$:

Some fcn's, however, are defined by a relation between x & y:

The first type of fcn's are called "explicit", & the second are "implicit". In some cases we can re-arrange an implicit fcn' to get an explicit one:

In some cases, we can not:

For fcn's where solving for y explicitly brings complications, we need a new way to find their derivatives - y' or $\frac{dy}{dx}$.
This consists of differentiating with respect to x & solving for y'.

"implicit differentiation"

In our cases so far, y was the dependent variable, & x was the independent variable.

Our variables do not always need to be called x & y. As long as we know which depends on which, we can use implicit differentiation to find derivatives.

ex// A population of bees (p) grows as time (t) goes on. The growth of its population can be described by the eqn:

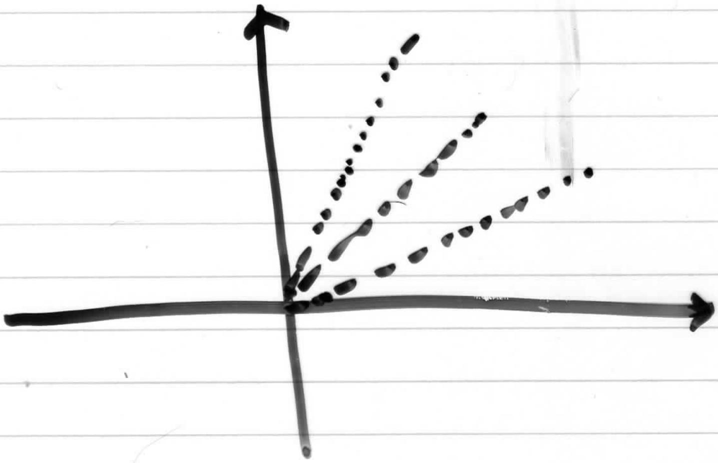
$$e^{pt} = t + p$$

Find the rate of change of the population.

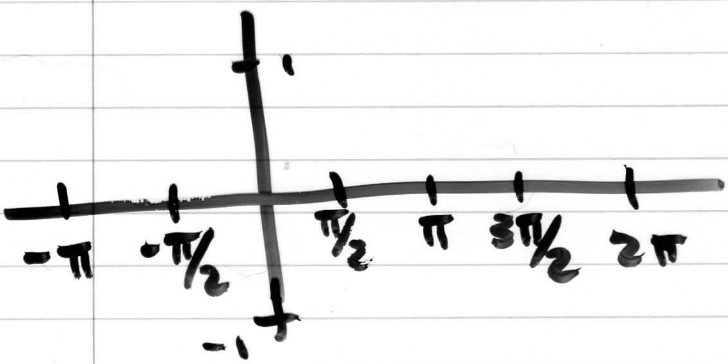
or, we can have more than one dependent variable...

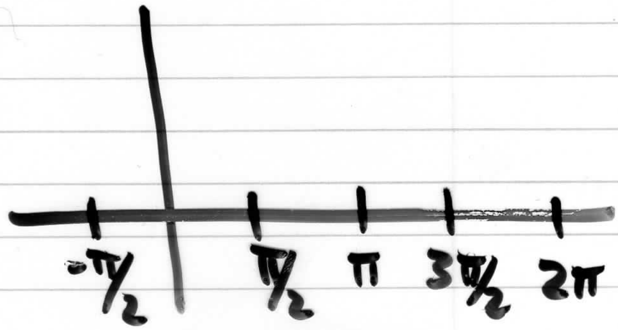
Section 3.9: Derivatives of Trig. Functions

Let's start with a review of trigonometry:



θ	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$ …	π …	$3\pi/2$ …	2π
$\sin\theta$								
$\cos\theta$								
$\tan\theta$								





identities:

Trig. fcn's are often used in modeling real-world phenomena. In particular; vibrations, waves, elastic motions, & other quantities that vary in a periodic manner.

ex/ An object at the end of a vertical spring is stretched 4cm beyond its rest position & released at time $t=0$. Its position at time t is $s=f(t)=4\cos t$.

Find the velocity at time t & use it to discuss the object's motion.

Steps For Derivatives

Which rule(s) do I need to use?

a) $\frac{d}{dx}(c)=0$

b) $\frac{d}{dx}(x^n)=nx^{n-1}$ (Power Rule)

c) Product Rule: $(fg)'=f'g+g'f$

d) Quotient Rule: $\left(\frac{f}{g}\right)'=\frac{f'g-g'f}{g^2}$

e) Trig Rules: $(\sin x)'=\cos x$, $(\cos x)'=-\sin x$

$(\tan x)'=\sec^2 x$, $(\csc x)'=-\csc x\cot x$

$(\sec x)'=\sec x\tan x$, $(\cot x)'=-\csc^2 x$

f) Chain Rule: $[f(g(x))]'=f'(g(x))\cdot g'(x)$

$\rightarrow [(g(x))^n]'=n(g(x))^{n-1}\cdot g'(x)$

Section 3.11: Derivatives of Exponentials & Logarithms

Review of inverse fcn's & logarithms:

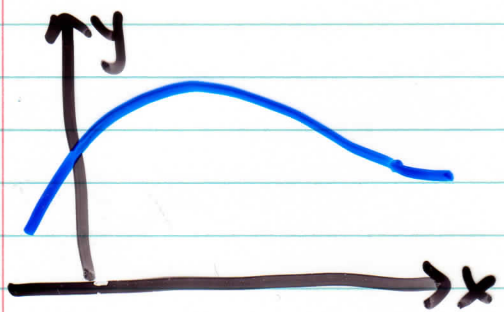
we can observe a population of bacteria & record pop. as a fcn' of time: $N = f(t)$, or, we could ask ourselves about the time it would take for the population to reach a certain level: $t = f^{-1}(N)$.

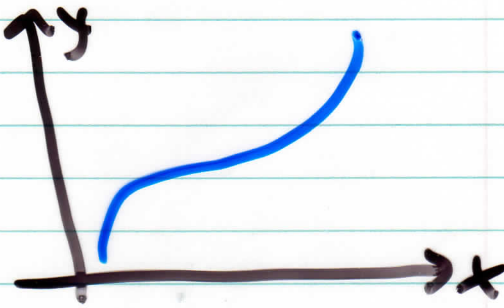
Not all fcn's have inverses:

Recall: To distinguish whether we have a fcn' → vertical line test.

To test if a fcn' is one-one, we can use the "horizontal line test".

ex// Is the following a 1-1 fcn?

a) 

b) 

c) $f(x) = x^2$

d) $f(x) = x^3$

Cancellation eqn's:

$f^{-1}(f(x)) = x$ & $f(f^{-1}(x)) = x$

ex// Find the inverse of $f(x) = x^3 + 2$.

The graph of f^{-1} is obtained by reflecting the graph of f about the line $y = x$:

ex// Sketch $f(x) = \sqrt{-1-x}$'s inverse:

Logarithmic Fcn's: If $a>0$ & $a\neq 1$, then the fcn' $f(x)=a^x$ is 1:1 & it's inverse is called the "logarithmic fcn' with base a".

If the base is e, we have the "natural logarithm"; $\ln(a)=\log_e a$.

Logarithm Laws: for $x>0$ & $y>0$:

1. $\log_a(xy)=\log_a x+\log_a y$
2. $\log_a\left(\frac{x}{y}\right)=\log_a x-\log_a y$
3. $\log_a(x^r)=r\log_a x$
4. $\log_a x=\frac{\ln x}{\ln a}$

Derivatives Rules for exp & log fcn's:

① $\frac{d}{dx}\left(\log_a(g(x))\right) = \frac{g'(x)}{g(x)\ln a}$

② $\frac{d}{dx}\left(\log_a x\right) = \frac{1}{x\ln a}$

③ $\frac{d}{dx}\left(\ln(g(x))\right) = \frac{g'(x)}{g(x)}$

④ $\frac{d}{dx}\left(\ln x\right) = \frac{1}{x}$

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① $\frac{d}{dx}\left(a^{g(x)}\right) = a^{g(x)} \cdot g'(x) \cdot \ln a$

② $\frac{d}{dx}\left(a^{x}\right) = a^{x} \cdot \ln a$

③ $\frac{d}{dx}\left(e^{g(x)}\right) = e^{g(x)} \cdot g'(x)$

④ $\frac{d}{dx}\left(e^{x}\right) = e^{x}$

Section 3.12: Logarithmic Differentiation

Sometimes, the calculation of derivatives of complex, layered fcn's can be made easier (via the Log Laws) by taking logarithms. Steps:

1. take natural logs (ln) of both sides of an eqn' of the form $y=f(x)$ & use the Log Laws to simplify.
2. Use implicit differentiation to take derivatives of both sides of the eqn'.
3. Solve for y'.
4. Substitute back $y=f(x)$.

Section 3.14: Rolle's Thm' & the Mean Value Thm'

In this section, we see how to tell whether a fcn' will have any points that satisfy $f'(c)=0$, & how to find a point $x=c$ where the slope of the tangent line = the slope of the secant line.

Rolle's Thm': Let f be a fcn that satisfies:

1) f is cts on the closed interval $[a,b]$

2) f is differentiable on (a,b)

3) $f(a)=f(b)$

Mean Value Thm: Let f be a fcn' that satisfies

1) f is cts on the closed interval $[a,b]$,
2) f is differentiable on (a,b),

Then there is a number c in (a,b) such that $f'(c) = \frac{f(b)-f(a)}{b-a}$

or $f(b)-f(a) = f'(c)(b-a)$.

ex// A taffic plane measures the time it takes a car to travel between points A & B as 15 s, and radios this info to a patrol car. What is the maximum speed at which the police officer can claim that the car was travelling between A & B if the distance between A & B is 500 m?