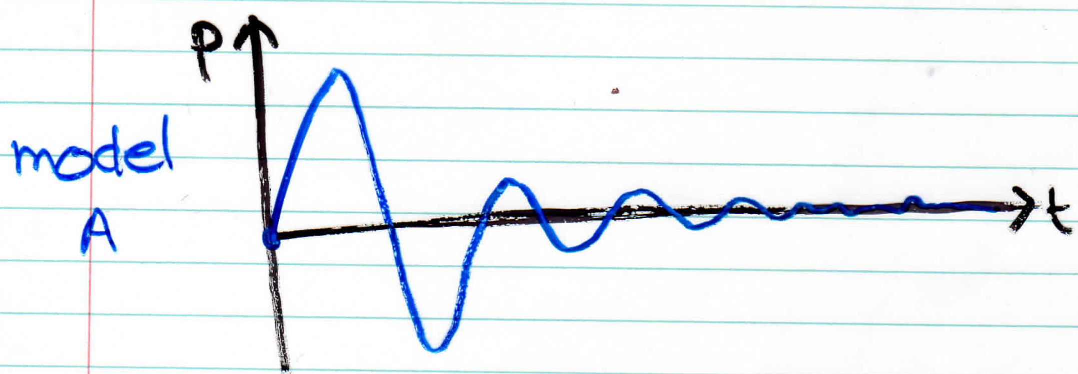
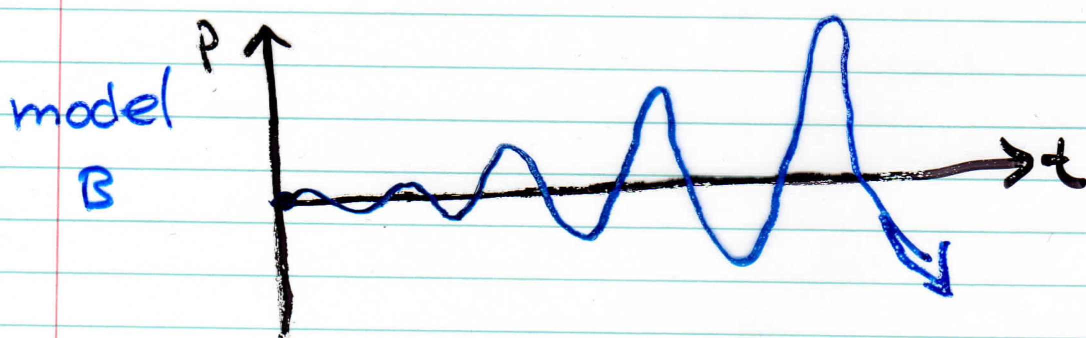
Chapter 2: Limits & Continuity

Scenario: You are commisioned to build a tall monument on an earthquake fault line. You build two models, & graph the fcn' representing each models' vibrations when some outside force is applied:



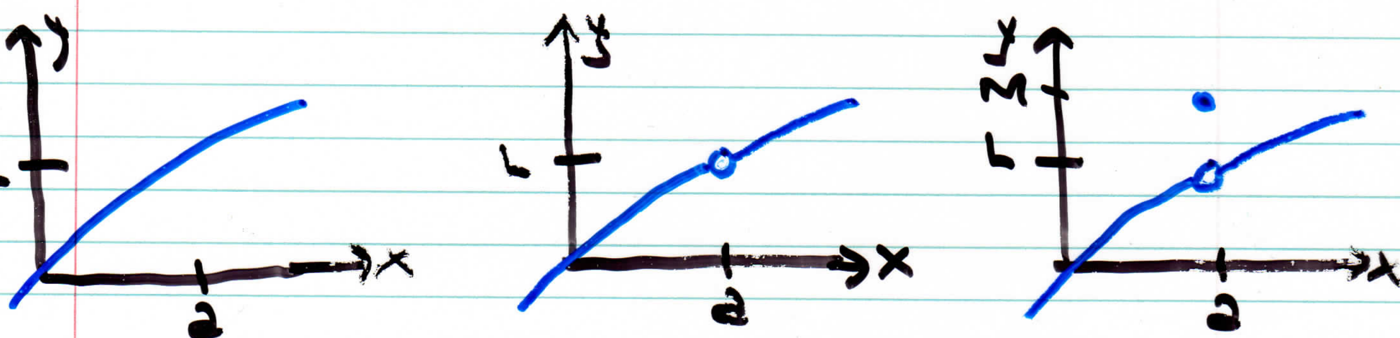


Section 2.1: Limits

What is the value of $f(x) = \frac{x-1}{x^2-1}$ at $x=1$?

Defn': $\lim_{x \to a} f(x) = L$ / the limit of $f(x)$ as x approaches a, equals L,

if we can make the values of $f(x)$ arbitrarily close to L by taking x to be close to a (on either side), but NOT EQUAL TO a



We can find limits by constructing tables of values, or by examining graphs.

ex// Examine the behaviour of the Heaviside Fcn'

$$H(t) = \begin{cases} 0, & \text{if } t < 0 \\ 1, & \text{if } t \geq 0 \end{cases} \quad \text{near } t = 0.$$

Defn': ① the limit of $f(x)$ as x approaches a from the LEFT is written $\lim_{x \to a^-} f(x) = L$.

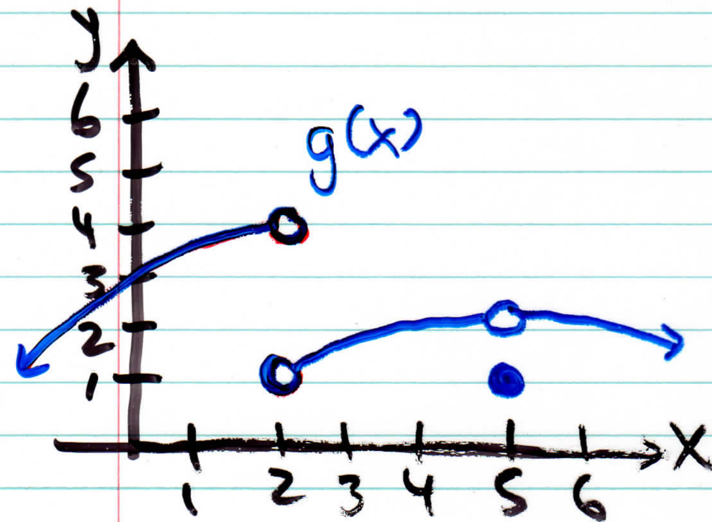
② the limit of $f(x)$ as x approaches a from the RIGHT is written $\lim_{x \to a^+} f(x) = L$.

Theorem: $\lim_{x \to a} f(x) = L$ if & only if

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L.$$

ex/ Use the graph of g to state the values (if they exist) of the following:

a) $\lim_{x \to 2^-} g(x) =$



b) $\lim_{x \to 2^+} g(x) =$

c) $\lim_{x \to 2} g(x) =$

d) $f(2)$

e) $\lim_{x \to 5^-} g(x) =$

f) $\lim_{x \to 5^+} g(x) =$

g) $\lim_{x \to 5} g(x) =$

h) $f(5) =$

LIMIT LAWS: Suppose c is a constant, and $\lim_{x\to a} f(x) = A$ & $\lim_{x\to a} g(x) = B$ (ie, both exist), then:

1. $\lim_{x\to a} [f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) = A+B$
2. $\lim_{x\to a} [f(x)-g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x) = A-B$
3. $\lim_{x\to a} [cf(x)] = c\lim_{x\to a} f(x) = cA$
4. $\lim_{x\to a} [f(x)g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x) = A\cdot B$
5. $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{A}{B}$ $(B \neq 0)$
6. $\lim_{x\to a} [f(x)]^n = [\lim_{x\to a} f(x)]^n = A^n$
7. $\lim_{x\to a} c = c$
8. $\lim_{x\to a} x = a$
9. $\lim_{x\to a} x^n = a^n$
10. $\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}$
11. $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)} = \sqrt[n]{A}$

Direct Substitution Property: If f is a polynomial or rational fcn' & a is in the domain of f, then:

$$\lim_{x\to a} f(x) = f(a)$$

Section 2.2: Infinite limits

For some fcn's the values of f(x) get very large (+ or −) at some values of x.

ex// find $\lim_{x \to 0} \frac{1}{x^2}$ & $\lim_{x \to 0} \frac{1}{x}$ (if they exist)

Defn: Let f be a fcn' defined on both sides of a, except at a itself, then $\lim_{x \to a} f(x) = \pm\infty$ means that the values of f(x) can be made arbitrarily large (+ or −) by taking x close to a, but not equal to a.

ex// determine the infinite limits:

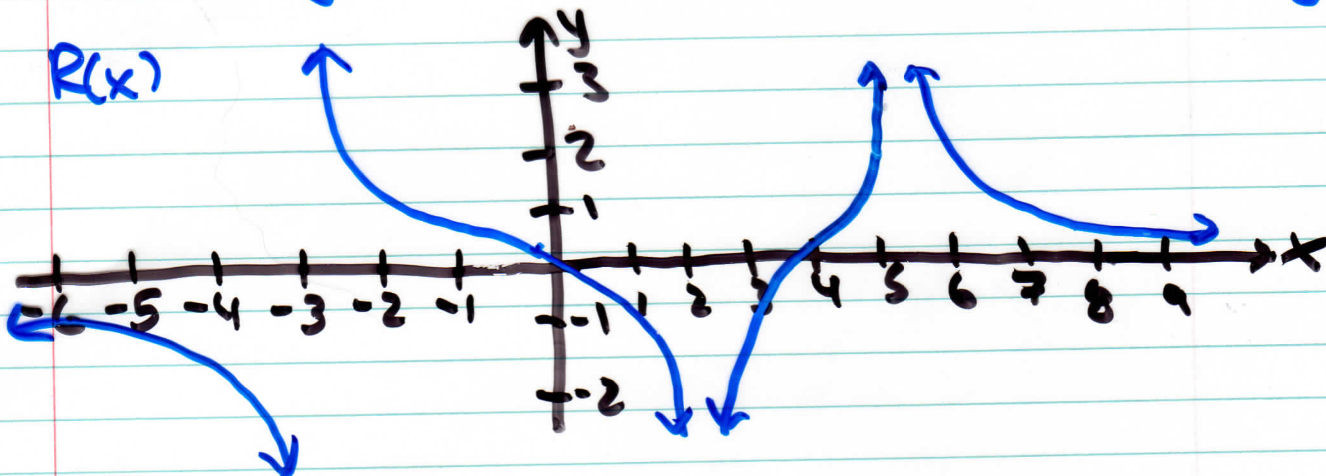
a) $\lim_{x \to 5} \frac{6}{x-5}$

b) $\lim_{x \to 1} \frac{2-x}{(x-1)^2}$

<u>Defn</u>': the line $x=a$ is a "vertical asymptote" (v.a.) if at least <u>ONE</u> of the following is true:

$\lim_{x\to a} f(x) = \pm\infty$, $\lim_{x\to a^-} f(x) = \pm\infty$, $\lim_{x\to a^+} f(x) = \pm\infty$

ex// For R(x) given below, state the following:



a) $\lim_{x\to 2} R(x) =$

b) $\lim_{x\to 5} R(x) =$

c) $\lim_{x\to -3} R(x) =$

d) the eqn's of any vertical asymptotes:

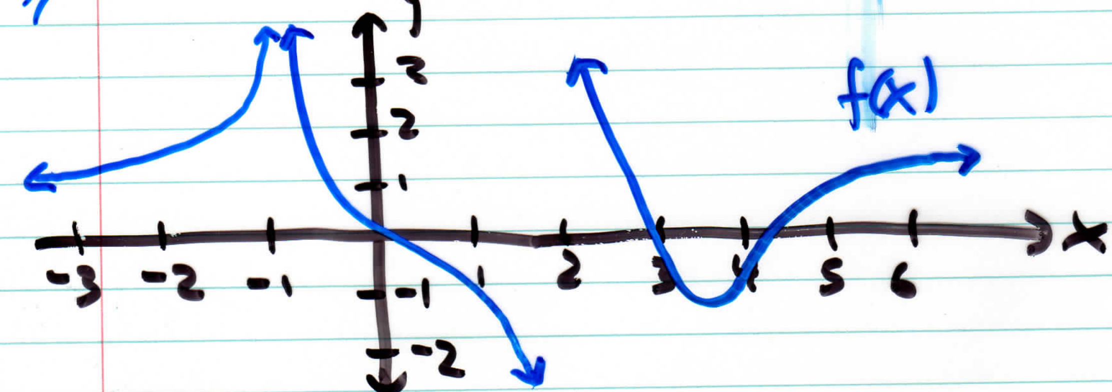
Section 2.3: Limits at Infinity

We already know that if $f(x)$ gets arbitrarily large at some x-value, we have a v.a.
Now, we let x become larger & larger (+ or −) & see what happens to $f(x)$.

For example, what happens to $\frac{1}{x}$ as $x \to \pm\infty$?

Defn': The line $y=L$ is a "horizontal asymptote" (h.a.) if either $\lim_{x\to\pm\infty} f(x) = L$.

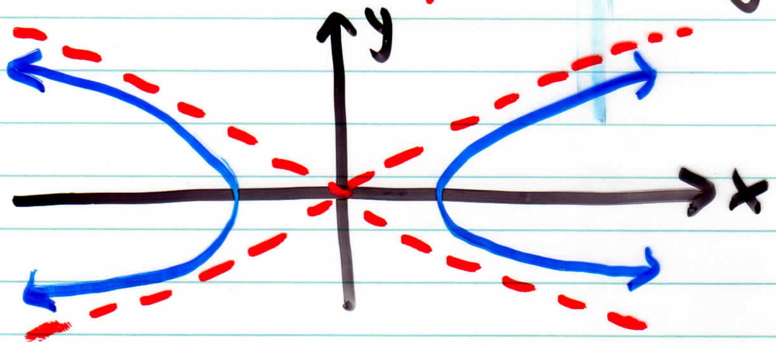
ex// Find the v.a.'s & h.a.'s of f(x):



Theorem': If $r>0$ is a rational number, then $\lim_{x\to\infty} \frac{1}{x^r} = 0$.

If $r<0$ is a rational number such that x^r is defined for all x, then $\lim_{x\to-\infty} \frac{1}{x^r} = 0$.

Hyperbolas have asymptotes, but not horizonal or vertical ones, rather they are "slanted" (or "oblique") asymptotes.



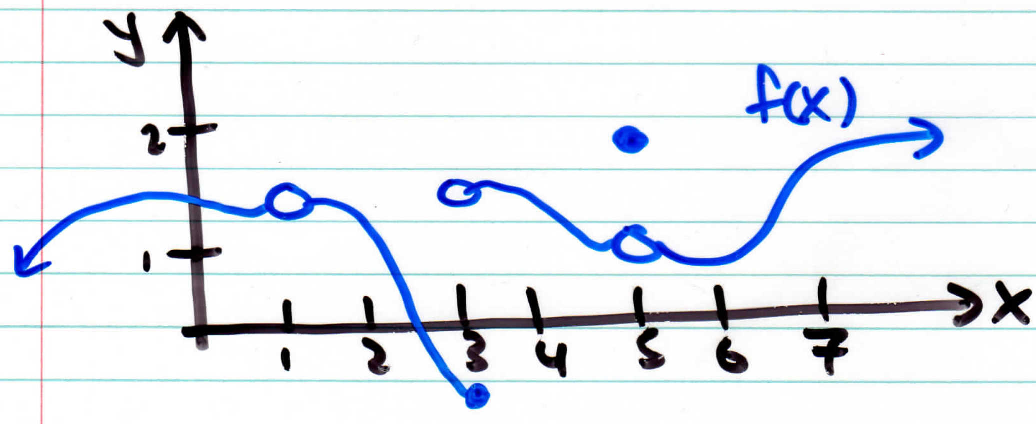
ex// Find all asymptotes of $f(x) = \dfrac{x^3 - 3x^2 + 1}{x^2 + 1}$

Section 2.4: Continuity

We noticed in the last section that $\lim_{x\to a} f(x)$ can often be found just by pluggin in a ($f(a)$). Functions with this property are called "continuous at a", ie/ $\lim_{x\to a} f(x) = f(a)$.

Otherwise, we say $f(x)$ is "discontinuous at a".
Geometrically, a continuous (cts) fcn' has a graph with no breaks in it.

ex// At which values of x is $f(x)$ discts? why?



Theorem ①: The following types of fcn's are cts. at every number in their domains

polynomials rational fcn's
root fcn's trig fcn'
exponential fcn's logarithmic fcn's

Theorem ②: If f & g are cts at a, & c is a constant, then the following fcn's are also cts. at a.

$f+g$ $f-g$ cf fg $\frac{f}{g}$

ex// Prove that if f & g are cts at a, then so if $f+g$.