Topic 3 Outline

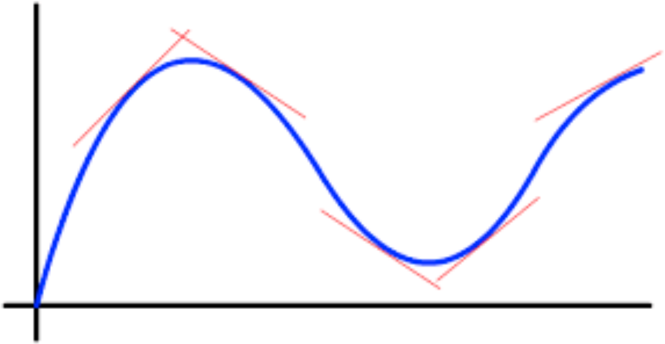
1. The Derivative
    - What is the Derivative?
    - Rates of Change
    - The Derivative as a Function
    - Differentiable $\Rightarrow$ Continuous

Topic 3 Learning Objectives

1. derive the derivative from the slope of the tangent representation
2. know the formal definition of the derivative, using both limit representations
3. calculate slope of the tangent
4. derive the derivative from the instantaneous rate of change representation
5. calculate instantaneous rate of change (velocity)
6. calculate derivatives using the definition of the derivative
7. graph $f'(x)$ from the graph of $f(x)$
8. prove that differentiable $\Rightarrow$ continuous

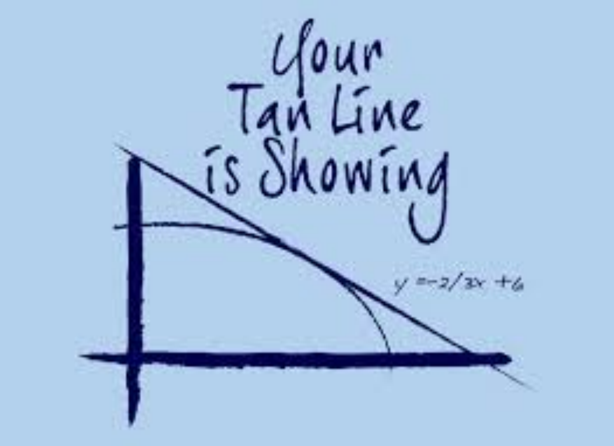
What is a Tangent Line?

A **tangent** to a curve at a point is a line that just touches the curve at that point:



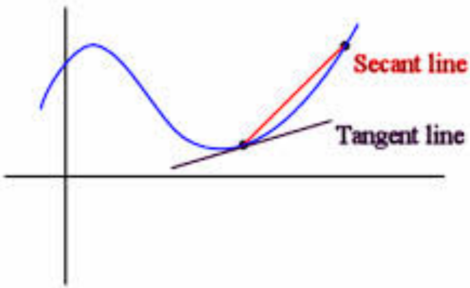
Check out this link for a video on the meaning of the derivative!
https://www.educreations.com/lesson/embed/9724142/?ref=app

What is a Tangent LIne?



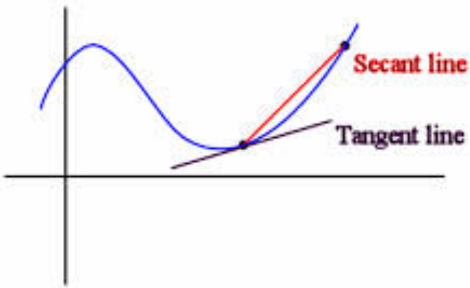
Slope of the Tangent

To figure out the slope of the tangent line at a point $(x, f(x))$ we begin by looking at the **secant** line:



Slope of the Tangent - Alternate Version

To figure out the slope of the tangent line at a point $(x, f(x))$ we begin by looking at the **secant** line:



Examples

1. Find the slope of the tangent line to the parabola $y = x^2$ at $x = 0$ and $x = -1$.

2. Find the equation of the tangent line to the curve $y = \frac{3}{x}$ at $x = 3$.

Rates of Change

If x changes from x_1 to x_2 and $y = f(x)$, then the change in x is $\delta x = x_2 - x_1$ and the corresponding change in y is $\delta y = f(x_2) - f(x_1)$
Using these ideas, we can calculate:

Average Rate of Change $\frac{\delta y}{\delta x} = \frac{f(x_2)-f(x_1)}{x_2-x_1}$

Instantaneous Rate of Change $\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{f(x_2)-f(x_1)}{x_2-x_1}$

One such rate of change is **velocity**: If an object moves along a straight line according to an equation of motion $s = f(t)$, where s is the displacement of the object at time t, then $f(t)$ is called the **position function**.

Rates of Change

Then, from time $t = a$ to time $t = a + h$:

Average Velocity $\frac{displacement}{time} = \frac{f(a+h)-f(a)}{h}$

As the time intervals get shorter and shorter (ie, as we let $h \to 0$), then the

Instantaneous Velocity (Velocity) at time $t = a$ $\lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$

Example

Suppose a ball is dropped from the upper observation deck of the CN Tower, 450m high, and the distance (in meters) falled after t seconds is $4.9t^2$. Then:

a What is the velocity of the ball after 5 seconds?

b How fast is the ball travelling when it hits the ground?

Derivatives

So, now we know that the slope of the tangent line to the curve $y = f(x)$ at the point where $x = a$ is given by:

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h},$$

and the velocity of an object with position finction $s = f(t)$ at time $t = a$ is given by:

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

In fact, this type of limit occurs widely in any of the sciences or engineering whenever we calculate a *rate of change*, so we give it a special name and notation:

The **derivative** of a function $f(x)$ at a number a is $f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ if this limit exists (or equivalently, $f'(a) = \lim_{x \to a} \frac{f(x)) - f(a)}{x - a}$.

Derivatives

Examples

1. Find the derivative if $f(x) = \frac{1}{x}$ at $x = a$ and $x = 4$.

2. Find $f'(a)$ if $f(x) = \sqrt{2x+1}$ at $x = a$ and $x = 4$.

3. A particle moves along a straight line with equation of motion $s = f(t) = t^2 - 6t - 5$ (t is in seconds, s is in meters). Find an expression for the velocity at time $t = a$, graph the velocity function, and find the time when the particle is at rest.

The Derivative as a Function

Recall, we defined the derivative of a function $f(x)$ at a *fixed* number a as:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h},$$

If we replace the fixed constant in the above equation by the variable x, then we can vary a!

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

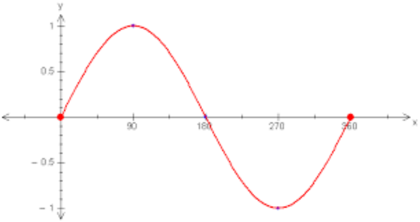
Now $f'(x)$ is a new function of x, the **derivative**!

Other notations for the derivative of $y = f(x)$ include:

$$f'(x) = \frac{df}{dx} = \frac{dy}{dx} = Df(x)$$

Example

Use the graph of $f(x)$ given below to sketch the graph of $f'(x)$



Examples

Find a fomula for $f'(x)$ and state its domain:

1. $f(x) = x^3 - x$

2. $f(x) = \sqrt{x-1}$

3. $f(x) = \frac{1-x}{2+x}$

Definition

A function is **differentiable at x=a** if $f'(a)$ exists. It is differentiable on an open interval ((a, b) or (a, ∞) , $(-\infty, b)$, $(-\infty, \infty)$) if it is differentiable at every number in the interval.

Take a look at the last example and state where those functions were differentiable.

In general, if the graph of a function $f(x)$ has a *corner* or a *kink*, then the graph has no tangent, and therefore no derivative, at that point. Look at the function $f(x) = |x|$:

Differentiable $\Rightarrow$ Continuous

Theorem: If $f(x)$ is differentiable at a, then $f(x)$ is continuous at a.

Proof:

Five in Five!

Solve the following in 5 minutes or less!

1. Find $f'(x)$ if $f(x) = 2x^2 + 1$
2. Find $f'(x)$ if $f(x) = \sqrt{9x}$
3. If an object is travelling with position function $s = f(t) = -(t-1)^2 - 1$, when will the object be moving fastest? slowest? (HINT: try graphing it!)
4. What is the slope of the tangent to the function $f(x) = 3x + 8$ at the point $x = 0$? $x = -1$? At any x-value?
5. Where is the function $f(x) = |x - 3|$ differentiable?

Flex the Mental Muscle!

Below are the images of 6 differently shaped water bottles. A mathematician is interested in how the water level of each is rising as time goes on, when he pours water into the bottle at a constant rate. Draw 2 rough sketches for each of the bottle shapes: one that represents how the height changes as he pours the water (h vs t), and the other that represents how the rate of change of the height changes as he pours the water (dh/dt vs t).