Topic 7 Outline

1. Maximum and Minimum Values
   - What Kind of Max and Min Values Does a Function Have?
   - The Extreme Value Theorem
   - Finding Absolute Extrema
   - The Mean Value Theorem

Topic 7 Learning Objectives

1. define absolute (global) and local (relative) extrema
2. state and use the Extreme Value Theorem (EVT)
3. find the absolute extrema of a function
4. state the Mean Value Theorem (MVT)
5. use the MVT to solve problems about max and min rates of change
6. prove that if $f'(x) = 0$ for all x in an interval (a, b), then f is constant on (a, b).

What Kind of Max and Min Values Does a Function Have?

Some of the most important applications of the calculus of derivatives involve finding the optimal way to do somthing. Sometimes, this means looking for a maximum, sometimes it means looking for a minimum.

Defn 1: A function $f(x)$ has an **absolute (global) max** at c if $f(c) \geq f(x)$ for all x values in D, where D is the domain of f. The number $f(c)$ is called the **maximum value** of f on D.

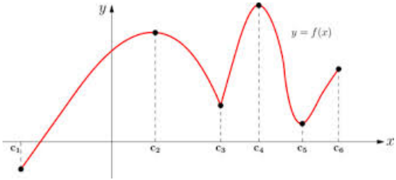
Similarly, a function $f(x)$ has an **absolute (global) min** at c if $f(c) \leq f(x)$ for all x values in D, and the number $f(c)$ is called the **minimum value** of f on D.

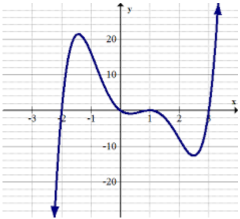
These maximum and minimum values are called the **extreme values** of f.

What Kind of Max and Min Values Does a Function Have?

Defn 2: A function $f(x)$ has a **local (relative) max** at c if $f(c) \geq f(x)$ when x is near c. Similarly, $f(x)$ has a **local (relative) min** at c if $f(c) \leq f(x)$ when x is near c

Example: Identify the local and absolute extrema in the graphs below:





Check out this link for a video on max and min values!
https://www.educreations.com/lesson/embed/9845848/?ref=app

Examples

What are the extreme values of the functions below?

1. $f(x) = x^2$

2. $f(x) = x^3$

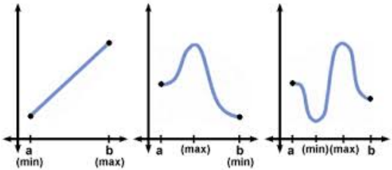
3. $f(x) = cosx$

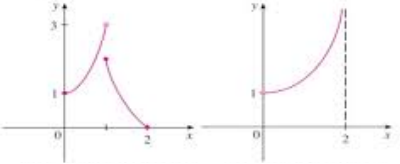
The Extreme Value Theorem

We have seen that some functions have extreme values, and others (like x^3) do not!

Extreme Value Theorem: If $f(x)$ is continuous on a closed interval $[a, b]$, then f attains an absolute max value of $f(c)$ at some $x = c$ and an absolute min value of $f(d)$ at some $x = d$ in $[a, b]$.

Example: Use the EVT to discuss the absolute extrema of the following functions:





The Extreme Value Theorem

The EVT says that a continuous function on a closed interval has an absolute mac value and an absolute min value, but not how to find these extreme values!

Fermat's Theorem: If $f(x)$ has a local max or min at $x = c$, and if $f(c)$ exists, then $f'(c) = 0$. BUT, we can not expect to locate extreme values simply by setting $f'(x) = 0$ and solving.

Why is that??

So Fermat's Theorem suggests that we should at least start looking for extreme values of f at the numbers $x = c$ where $f'(c) = 0$ or $f'(c) dne$! We call these numbers (c's) the **critical numbers** or **critical points** (cp's) of the function.

Examples

Find the critical numbers of the following functions:

1. $f(x) = x^3 + +x^2 + x$
2. $f(x) = x^{\frac{3}{5}}(4 - x)$

Find the absolute extrema of the following functions:

1. $f(x) = x^3 + +x^2 + x$ on $[-2, 2]$
2. $f(x) = x^3 + +x^2 + x$
3. $f(x) = \frac{1}{x^2-4}$
4. $f(x) = \frac{1}{x^2-4}$ on $[-1, 1]$

Finding Extreme Values

So, to find the extreme values of a function that is continuous on a closed interval, we must:

1. Find the critical numbers of the function,
2. Find the values of the function at those critical numbers,
3. Find the values of the function at the endpoints ,
4. compare the values from steps 2 and 3 to identify the highest and lowest as our extreme values.

If the function is not continuous, we should look for any vertical asymptotes to decide whether the function has absolute extreme values.

If it is not defined on a closed interval, we should examine the limits at $\pm\infty$ or the limits as we approach the endpoints to decide whether the function has absolute extreme values.

The Mean Value Theorem

Now we know that at the number c where $f'(c) = 0$ we may have a max or min value. In this section we see how to tell whether a function will have *any* points that satisfy $f'(c) = 0$ without actually calculating the derivative!

Rolle's Theorem: Let f be a function that satisfies the following 3 properties:

1. f is continuous on the closed interval $[a, b]$
2. f is differentiable on (a,b)
3. $f(a) = f(b)$

Then there exists a number c in (a, b) such that $f'(c) = 0$.

Apply Rolle's Theorem to a ball you throw straight up in the air:

The Mean Value Theorem

Now we know that at the number c where $f'(c) = 0$ we may have a max or min value. In this section we see how to tell whether a function will have *any* points that satisfy $f'(c) = 0$ without actually calculating the derivative!

Rolle's Theorem: Let f be a function that satisfies the following 2 properties:

1. f is continuous on the closed interval $[a, b]$
2. f is differentiable on (a,b)

Then there exists a number c in (a, b) such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

This tells us that there is some point c in $(a, b)$where the slope of the tangent line is equal to the slope of the secant line!

Examples

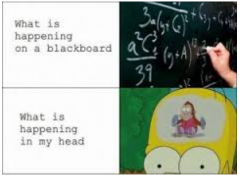
1. Suppose $f(0) = -3$ and $f'(x) \leq 5$ for all values of x in the domain of a continuous function. How large can $f(2)$ be?

2. Police lose their radar gun, and instead set up two stations on a highway at points A and B 10km apart to catch speeders. If the speed limit is 60km/h and you are clocked at point A at 12:00pm and point B at 12:05pm, can the police give you a ticket for speeding?

Mean Value Theorem

Theorem: If $f'(x) = 0$ for all x in an interval (a, b), then f is constant on (a, b).

Proof:



Five in Five!

1. Identify the absolute and local extrema of the function $x^2 + 4$.

2. Find the critical points of the function $x^3 - 3x^2 + 1$.

3. Find the absolute extrema of the function $x^3 - 3x^2 + 1$, or state why they do not exist.

4. Find the absolute extrema of the function $x^3 - 3x^2 + 1$ on the interval $[-1, 4]$, or state why they do not exist.

5. Find the absolute extrema of the function $\frac{2x-1}{3+x}$ on the interval $[-1, 4]$, or state why they do not exist.

Flex the Mental Muscle!

For the FALSE statements below, show that they are false by drawing a picture that is a counterexample.

1. If a function is continuous, it will have both an absolute max and absolute min.
2. If a function is NOT continuous, it will not have an absolute max or an absolute min.
3. A function that is defined on a closed interval will always have an absolute max or an absolute min.
4. The absolute max value of a function must always be greater than the absolute min value of a function.
5. A function that is continuous and defined on a closed interval will have a finite number of absolute maxima and absolute minima.