Topic 2 Outline

1. Limits and Continuity
   - What is a Limit?
   - Calculating Limits
   - Infinite Limits
   - Limits at Infinity
   - Continuity

Topic 2 Learning Objectives

1. know the formal definition of a limit, and the overall concept of what a limit represents
2. calculate a variety of types of limits, visually, numerically, or algebraically by:
   - plugging in
   - factoring
   - rationalizing
   - combining parts
   - by inspection (graphing)
   - squeeze theorem
3. solve infinite limits
4. locate equations of vertical asymptotes of functions
5. solve limits at infinity
6. locate equations of horizontal asymptotes of functions
7. define continuity both visually and by the defintion (3 Rules)
8. solve problems using the definition of continuity

What is a Limit?

Lets begin our analysis of limits by looking at an example.
What is the value of the function $f(x) = \frac{x-1}{x^2-1}$ at $x = 1$?

What about at values close to $x = 1$?

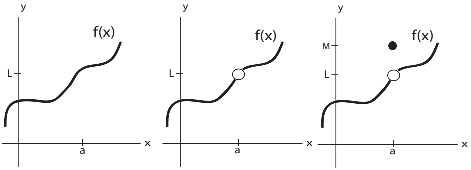
x					
$f(x)$					

Definition

If we can make the values of a function $f(x)$ arbitrairly close to some output value L by taking x to be close to some input value a (on either side) but not equal to a, then we say that the **limit** of $f(x)$ as x approaches a is equal to L:

$$\lim_{x \to \infty} f(x) = L$$

Check out this link for a video on the defintion of a limit!
https://www.educreations.com/lesson/embed/9666683/?ref=app
Look at the following 3 functions:



Example

The Heaviside Function is defined as:

$$H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$$

What is

$$\lim_{t \to 0} H(t) = ?$$

Definition

1. The limit of $f(x)$ as x approaches from the **left** is written as:

$$\lim_{x \to a^-} f(x)$$

2. The limit of $f(x)$ as x approaches from the **right** is written as:

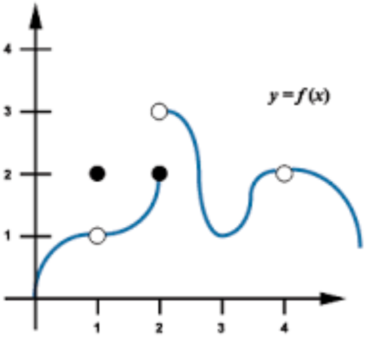
$$\lim_{x \to a^+} f(x)$$

Theorem: $\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a^-} = \lim_{x \to a^+} = L$.

Example

Using the graph below, find the values (if they exist) of:

1. $\lim_{x \to 1^-} f(x) =$
2. $\lim_{x \to 1^+} f(x) =$
3. $\lim_{x \to 1} f(x) =$
4. $f(1) =$
5. $\lim_{x \to 2^-} f(x) =$
6. $\lim_{x \to 2^+} f(x) =$
7. $\lim_{x \to 2} f(x) =$
8. $\lim_{x \to 4} f(x) =$
9. $f(4) =$



Example

Sketch the following piecewise defined function and use it to determine the values for which $\lim_{x\to a} f(x)$ exists.

$$f(x) = \begin{cases} 2-x & \text{if } x < -1 \\ x-1 & \text{if } -1 \leq x < 1 \\ (x-1)^2 & \text{if } x \geq 1 \end{cases}$$

Check out this link for a video on limits and piecewise functions!
https://www.educreations.com/lesson/embed/9700871/?ref=app

Calculating Limits

Rather than using graphs and/or tables of values to estimate limits, we can calculate the values of limits algebraically by using the Limit Laws:

1. $\lim_{x\to a}[f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)]$
2. $\lim_{x\to 1}[cf(x)] = c \lim_{x\to 1} f(x)$
3. $\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x) \lim_{x\to a} g(x)]$
4. $\lim_{x\to a}[\frac{f(x)}{g(x)}] = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ as long as $\lim_{x\to a} g(x)$ exists.
5. $\lim_{x\to 1}[f(x)]^n = [\lim_{x\to 1} f(x)]^n$
6. $\lim_{x\to a} c = c$
7. $\lim_{x\to a} x^n = a^n$
8. $\lim_{x\to a} \sqrt{n}x = \sqrt{n}a$

Direct Substitution Property: If $f(x)$ is a polynomial or rational function and a is in the domain of $f(x)$, then $\lim_{x\to a} f(x) = f(a)$.

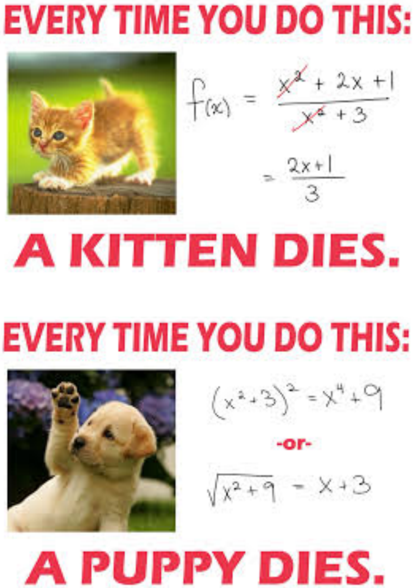
Examples

Evaluate the following limits:

1. $\lim_{x\to 5}(2x^2 - 3x + 4) =$
2. $\lim_{x\to 0}\sqrt{x} =$
3. $\lim_{x\to 1}\frac{x-1}{x^2-1} =$
4. $\lim_{t\to 0}\frac{3+t)^2-9}{t} =$
5. $\lim_{h\to 0}\left(\frac{9}{h(h+3)} - \frac{3}{h}\right) =$
6. $\lim_{x\to 1}\frac{|x-1|}{x^2-x} =$
7. $\lim_{t\to 0}\frac{\sqrt{t^2+9}-3}{t^2} =$

Check out this link for a video on calculating limits!
https://www.educreations.com/lesson/embed/9670681/?ref=app

WARNING!


EVERY TIME YOU DO THIS:
$f(x) = \frac{x^2 + 2x + 1}{x^2 + 3}$
$= \frac{2x+1}{3}$
A KITTEN DIES.
EVERY TIME YOU DO THIS:
$(x^2+3)^2 = x^4+9$
-or-
$\sqrt{x^2+9} = x+3$
A PUPPY DIES.

Squeeze Theorem

If these techniques fail, and particularly if we are working with trig functions, another thing to try is **Squeeze Theorem**.

Squeeze Theorem: If $f(x) \leq g(x) \leq h(x)$ when x is near a (not at a), and $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$, that implies that $\lim_{x \to a} g(x) = L$ also!

Example: Show that $\lim_{x \to 0} x^2 sin(\frac{1}{x}) = 0$

Infinite Limits

For some functions, the values of $f(x)$ get very large (positively or negatively) at certain values of x. Lets analyze this by looking at an example: Find $\lim_{x \to 0} \frac{1}{x}$ if it exists.

What about looking at values close to $x = 0$?

x							
$f(x)$							

Definition

Let $f(x)$ be a function defintied on both sides of a, except at a itself. Then $\lim_{x \to a} f(x) = \pm\infty$ means that the values of $f(x)$ can be made arbitrarily large (positively or negatively) by taking x close to a.

Look at the following examples:

1. $\lim_{x \to 5} \frac{6}{x-5} =$

2. $\lim_{x \to 1} \frac{2-x}{(x-1)^2} =$

Infinite Limits

After explaining to a student through various lessons and examples that:

$$\lim_{x \to 8} \frac{1}{x-8} = \infty$$

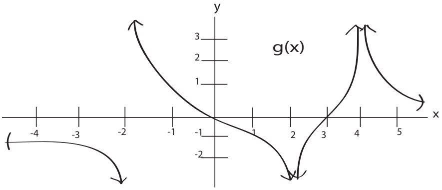
I tried to check if she really understood that, so I gave her a different example. This was the result:

$$\lim_{x \to 5} \frac{1}{x-5} = \rotatebox{90}{5}$$

Infinite Limits

The line $x = a$ is called a **vertical asymptote** if at least ONE of the following is true:
$\lim_{x \to a} f(x) = \pm\infty$ or $\lim_{x \to a^-} f(x) = \pm\infty$ or $\lim_{x \to a^+} f(x) = \pm\infty$.
Example: For the function g(x) shown below, use the graph to state the following:

1. $\lim_{x \to 4} g(x) =$
2. $\lim_{x \to 2} g(x) =$
3. $\lim_{x \to -2^-} g(x) =$
4. $\lim_{x \to -2^x} g(x) =$
5. the equations of any vertical asymptotes



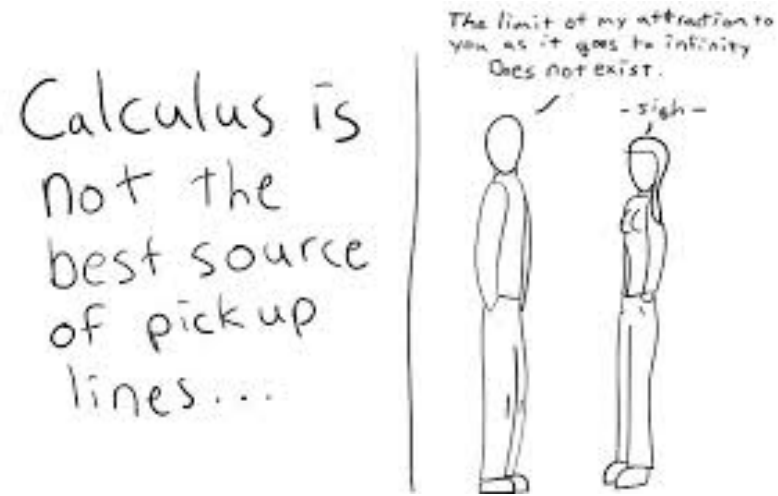
Limits at Infinity

We already know what a vertical asymptote is, and how to find one. Now, we let x become larger and larger (positively or negatively) and see what happens to our outputs (y-values). Look at $\lim_{x \to \infty} \frac{1}{x}$ and $\lim_{x \to -\infty} \frac{1}{x}$:

What happens to the outputs as our x values get larger and larger?

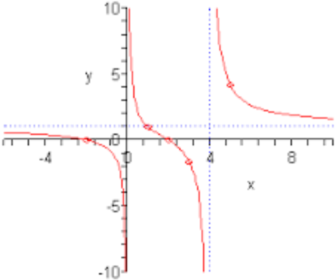
x							
$f(x)$							

Limits at Infinity



Definition

The line $y = a$ is called a **horizontal asymptote** if either:
$\lim_{x \to -\infty} f(x) = L$ or $\lim_{x \to \infty} f(x) = L$.
Example: For the function g(x) shown below, use the graph to find the infinite limits (vertical asymptotes) and limits at infinity (horizontal asymptotes):



Check out this link for a video on limits at and to infinity!
https://www.educreations.com/lesson/embed/9703534/?ref=app

Theorem

If $r > 0$ is a rational number, then $\lim_{x\to\infty} \frac{c}{x^r} = 0$.
If $r < 0$ is a rational number such that x^r is defined for all x, then $\lim_{x\to-\infty} \frac{c}{x^r} = 0$.
Solve the following:

1. $\lim_{x\to\infty} \frac{3x^2-x-2}{5x^2+4x+1}$
2. $\lim_{x\to-\infty} \frac{x^3+5x}{2x^3-x^2+4}$
3. $\lim_{x\to\infty} \frac{x^2+x}{3+2x}$
4. $\lim_{x\to-\infty} \frac{x^3}{x+1}$
5. $\lim_{x\to\infty} \frac{\sqrt{x^2+1}-x}{1}$
6. $\lim_{x\to-\infty} \frac{\sqrt{x^2+7x}}{x-3}$
7. $\lim_{x\to 0} e^{\frac{1}{x}}$
8. $\lim_{x\to\infty} sinx$

Continuity

We noticed in the last sections that for some functions, we can find $\lim_{x \to a} f(x)$ just by plugging in. Functions with this property are called **continuous at x=a**.

Specifically, this means that $\lim_{x \to a} f(x) = f(a)$

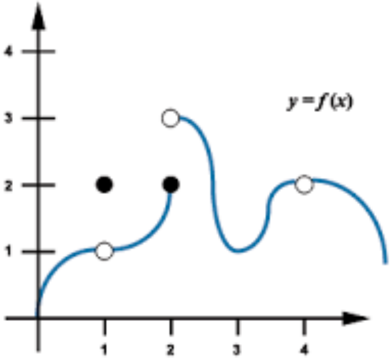
Otherwise, we say that the function is **discontinuous at x=a**.

Check out this link for a video on continuity!

https://www.educreations.com/lesson/embed/9704910/?ref=app

Example

At which numbers is $f(x)$ shown below discontinuous? Why?



Theorems

Theorem 1: If $f(x)$ and $g(x)$ are continuous at a and c is a constant, then the following are also continuous at a:

1. $f(x) + g(x)$
2. $f(x) - g(x)$
3. $cf(x)$
4. $f(x)g(x)$
5. $\frac{f(x)}{g(x)}$ (if $g(a) \neq 0$)
6. $f(x) \circ g(x)$ (if $f(x)$ is continuous at $g(a)$)

Theorem 2: The following types of functions are continuous at every number in their domains:

1. polynomials
2. rational functions
3. root functions
4. trig functions
5. exponential functions
6. logarithmic functions

Example

Where are the following discontinuous? Why?

1. $f(x) = \frac{x^2-x-2}{x-2}$

2. $$f(x) = \begin{cases} \frac{x^2-x-2}{x-2} & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases}$$

3. $$f(x) = \begin{cases} \frac{1}{x} & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}$$

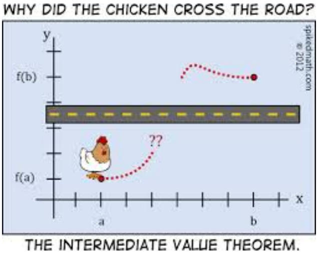
Example

Find a constant c that makes the function $g(x)$ coninuous everywhere if

$$g(x) = \begin{cases} c^2 + cx & \text{if } x < 5 \\ 2xc - 6 & \text{if } x \geq 5 \end{cases}$$

Intermediate Value Theorem

Suppose $f(x)$ is continuous on the closed interval $[a, b]$, and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$.



This can be useful in finding roots of a function!
Example: Show that there is a root of the equation
$f(x) = 4x^3 - 6x^2 + 3x - 2 = 0$ between $x = 1$ and $x = 2$.

Five in Five!

Solve the following in 5 minutes or less!

1. $\lim_{x\to 1^-} \frac{x^4-1}{x-1}$

2. $\lim_{x\to 2} \frac{|x-2|}{x^2+x-6}$

3. $\lim_{x\to 4} \frac{x^2-25}{4-x}$

4. $\lim_{x\to \infty} \frac{x^2-x-6}{3x^2-7}$

5. Is the function below continuous at $x = 0$?

$$f(x) = \begin{cases} -x^2+1 & \text{if } x \leq 0 \\ cosx & \text{if } x > 0 \end{cases}$$

Flex the Mental Muscle!

1. The 4 statements below are FALSE. Provide a counterexample (in the form of an equation of a function or a sketch of a function) for each.
   1. A function never reaches a limit value (ie, if the limit as x approaches a is equal to L, then the function value at a can not equal L).
   2. To evaluate the limit for any type of function, you should try to plug in the x-value and if you can plug it in without getting an undefined output then this is the answer.
   3. A piecewise function that is made up of two pieces, one on $x < 1$ and the other on $x \geq 1$ will have an undefined limit as x approaches 1.
   4. If, for a function $f(x)$, $f(2)$ is undefined, then that means that the limit of $f(x)$ as x approaches 2 is also undefined.

2. Design a function that has:
   1. Two DIFFERENT horizontal asymptotes. State the function, make a rough sketch, and solve the associated limits.
   2. Vertical asymptotes at $x = 3$ and $x = -3$. State the function, make a rough sketch, and solve the associated limits.