Topic 4 Outline

1. Derivative Rules
   - Calculating the Derivative Using Derivative Rules
   - Implicit Functions
   - Higher-Order Derivatives

Topic 4 Learning Objectives

1. calculate the derivative of:
   - polynomials and basic exponentials
   - products
   - quotients
   - trig functions
   - composite functions
2. prove that $[f(x) \pm g(x)]' = f'(x) \pm g'(x)$
3. prove that $[f(x)g(x)]' = f'(x)g(x) + g'(x)f(x)$
4. prove that $[sinx]' = cos(x)$
5. distinguish when and how to use each of the rules above, including combinations of them
6. calculate the derivative of implicit functions
7. calcuate higher-order derivatives

Derivative Rules

We can find derivatives in a faster way than using the limit definition of the derivative, which can be tedious and nearly impossible even for simple functions!

Check out this link for a video on the shortcut derivative rules!
https://www.educreations.com/lesson/embed/9725600/?ref=app

Derivatives of Polynomials and an Exponential

1. $\frac{d}{dx}(c) =$
2. $\frac{d}{dx}(x^n) =$ Find the derivatives of the following:
   1. $f(x) = x^6$
   2. $f(x) = x^1 00$
   3. $f(x) = \frac{1}{x}$
   4. $f(x) = \sqrt{x}$
   5. $f(x) = \sqrt[3]{x^4}$
3. $\frac{d}{dx}(e^x) =$
4. What is the slope of the tangent line to the curve $y = e^x$ at $x = 0$?

Derivatives of Polynomials and Exponentials

Derivative Laws

There are also two basic laws for calculating derivatives, they say that:

- $\frac{d}{dx}[cf(x)] = c\frac{d}{dx}[f(x)] = cf'(x)$
- **Proof**:

- $\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x) = f'(x) \pm g'(x)$
- **Proof**:

Examples

Find the derivatives of the following:

1. $f(x) = 186.5 + \pi$
2. $y = 3e^x + e^2$
3. $g(t) = \frac{4}{\sqrt{t}} + (\frac{1}{2}t)^5$
4. $p(r) = \frac{r^2+4r+3}{\sqrt{r}}$
5. Find the points on the curve $y = x^4 - 6x^2 + 4$ where the tangent line is horizontal.

The Product Rule

If the derivative law tells us that $[f(x) \pm g(x)]' = f'(x) \pm g'(x)$, we might also assume that $[f(x)g(x)]' = f'(x)g'(x)$.
Is this true?? Let's check using $f(x) = x$ and $g(x) = x^2$...

The Product Rule

The Product Rule: $[f(x)g(x)]' = f'(x)g(x) + g'(x)f(x)$
Proof:

The Quotient Rule

If the derivative law tells us that $[f(x) \pm g(x)]' = f'(x) \pm g'(x)$, we might also assume that $[\frac{f(x)}{g(x)}]' = \frac{f'(x)}{g'(x)}$.
Is this true?? Let's check using $f(x) = x$ and $g(x) = x^2$...

The Quotient Rule: $[\frac{f(x)}{g(x)}]' = \frac{f'(x)g(x) - g'(x)f(x)}{g(x)^2}$

Examples

Find the derivatives of the following:

1. $f(x) = xe^x$

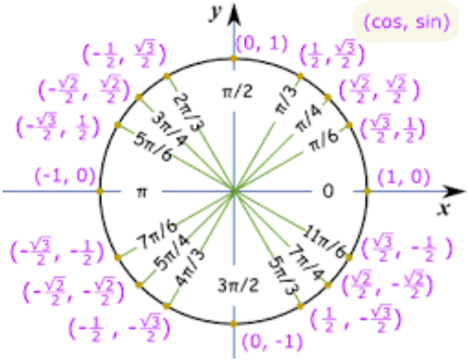
2. $y = \frac{3x^2+2\sqrt{x}}{x}$

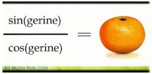
3. $g(t) = \frac{t^3e^t}{3t+t^e}$

4. Suppose $f(5) = 1$, $f(5) = 1$, $f(5) = 1$ and $f(5) = 1$, find
   1. $(f+g)'(5)$
   2. $(fg)'(5)$
   3. $(\frac{g}{f})'(5)$

Review of Trig. Functions

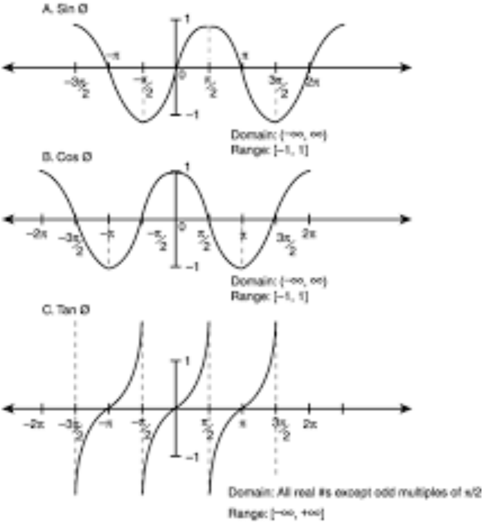
The major trig functions are: *sinx*, *cosx*, and *tanx* , and we will also look at the other three functions *cscx* , *secx* , and *cotx* .
This would be a good time to check out LevelUp on your UMLearn page!
The Unit Circle:





Review of Trig. Functions

Trig. Graphs:



Trig. Limits and Identities

Trig. Identities:

- $sin^2x + cos^2x = 1$
- $sin(A+B) = sinAcosB + sinBcosA$
- $cos^2(x) = \frac{1}{2}(1 + cos(2x))$
- $sin^2(x) = \frac{1}{2}(1 - cos(2x))$

Trig. Limits:

- $\lim_{x \to 0} \frac{sinx}{x} = 1$
- $\lim_{x \to 0} \frac{cosx - 1}{x} = 0$

Example: Calculate $\lim_{x \to 0} xcotx$

Derivatives of Trig. Functions

Use the graph of $f(x) = sinx$ to graph the function $sin'(x)$

Derivatives of Trig. Functions

Now we see that if $f(x) = \sin x$, then $f'(x) = \cos x$!
Proof:

Derivative Rules

Use what we have learned so far to find the derivatives of the rest of the trig. functions:

1. $f(x) = cosx$

2. $f(x) = tanx$

3. $f(x) = cscx$

4. $f(x) = secx$

5. $f(x) = cotx$

Examples

Find the derivatives of the following:

1. $f(x) = x - 3sinx$

2. $g(t) = \frac{4cost}{sect - 2tant}$

3. Find the equation of the tangent line to the curve $y = x^2 + sinx + 1$ at the point where $x = \frac{\pi}{2}$.

Derivatives of Trig. Functions

Trig. functions are often used in modeling real world phenomena. In particular, vibrations, waves, elastic motions, and other quantities that vary in a periodic manner.

Example: An object at the end of a verticaal spring is stretched 4cm beyond its rest position and released at time $t = 0$. Its position at any time t is given by $s = f(t) = 4cost$. Find the velocity of the object at time t and use it to analyze the object's motion.

The Chain Rule

So far, we can calculate the derivatives of most functions (polynomials, sums, differences, products, quotients...). However, we have not yet seen how to find the derivative of a function that is *inside* another function - a composite function!

Examples:

So, if f and g are both differentiable and $F = f \circ g$ is the composite function defined by $F(x) = f(g(x))$, then F is differentiable and the **Chain Rule** is:

$$[f(g(x))]' = F'(x) = f'[g(x)]g'(x)$$

Examples

Differentiate:

1. $f(x) = \frac{2}{x+1}$

2. $g(t) = \sqrt{5e^x + 1}$

3. $g(t) = sin(x^2)$

4. $g(t) = sin^2 x$

Steps for Derivatives

1. Which rule(s) do I need to use?
   1. $\frac{d}{dx}(c) = 0$
   2. $\frac{d}{dx}(x^n) = nx^{n-1}$
   3. $\frac{d}{dx}(e^x) = e^x$
   4. Product Rule $(fg)' = f'g + g'f$
   5. Quotient Rule $(\frac{f}{g})' = \frac{f'g - g'f}{g^2}$
   6. Trig. Rules $\frac{d}{dx}(sinx) = cosx$, $\frac{d}{dx}(cosx) = -sinx$, $\frac{d}{dx}(tanx) = sec^2x$, $\frac{d}{dx}(cscx) = -cscxcotx$, $\frac{d}{dx}(secx) = secxtanx$, $\frac{d}{dx}(cotx) = -csc^2x$
   7. Chain Rule $[f(g(x))]' = f'[g(x)]g'(x)$
2. Start with the "Big Picture" rules first, then work your way inside!

Examples

Let's put everything together to work through some more complex examples:

1. $f(x) = (\frac{x-2}{2x+1})^2$

2. $g(t) = (3t + \pi)^4(t^7 - t - 9)^5$

3. $h(t) = e^{sint}$

4. $p(t) = \sqrt[3]{1 + tan^2 t}$

Implicit Differentiation

So far, all of the functions we have been working with have been of the type $y = f(x)$, where we have one variable in terms of another:

ie: $y = \frac{1}{x}$, $y = 3x^2 + 1$, $y = sinxcosx$...

Some functions, however, are defined by a *relation* between x and y:

ie: $x + y = 25$, $sin(xy) = \frac{1}{xy} = x + 2y + 1$...

The first type of functions are called **explicit**, and the second are called **implicit**. In some cases we can rearrange an implicit function to get an explicit one, but in other cases we can not:

Check out this link for a video on implicit differentiation!
https://www.educreations.com/lesson/embed/9748304/?ref=app

Implicit Differentiation

For these types of functions or functions where solving for y brings complications, we need a new way to find their derivatives. This consists of differentiating both sides of the equation and then solcing for $y'(x)$, and is called **Implicit Differentiation**.

Example: If $x^2 + y^2 = 25$, find $\frac{dy}{dx}$:

Examples

Find $\frac{dy}{dx}$:

1. $x^3 + 3xy = 17$

2. $sin(x + 2y) = \frac{x^2}{cosy}$

3. $h(t) = e^{sint}$

4. $\sqrt{y} + e^x = e^y + 1$

5. Find the slope of the tangent line to the curve $x^2 + xy + y^3 = 3$ at the point $(1, 1)$.

Implicit Differentiation

In our cases so far, y was the *dependent* variable and x was the *independent* variable. Our variables do not always need to be called y and x, as long as we know which depends on which, we can use implicit differentiation to find the derivatives of any variables and any number of variables!

Example: Suppose a circle expands as time goes on, at what rate do area and radius change with respect to time?

Higher Order Derivatives

If f(x) is a differentiable function, then its derivative is also a function, and so may have derivatives of its own!

Other notations:

Examples

1. For $f(x) = x^5 + 3^4 + 9x^3 - x^2 - x + 7$, find $f'''(x)$ and $f^{(10)}(x)$.

2. Find $\frac{d^{27}}{dx^{27}}(cos x)$

Higher Order Derivatives

In general, we can interpret the second derivative as a rate of change of a rate of chane. If the first derivative is *velocity*, then its derivate (the second derivative) represents *acceleration*.

Example: The position of a particle is given by the equation of motion $s = f(t) = t^3 - 6t^2 + 9t$ (where t is in seconds and s is in meters). Find the accelearation at time t. What is the acceleration after 4 seconds? Graph all three functions together and discuss the particles *speed*.

Five in Five!

Solve the following in 5 minutes or less!

1. Find $f'(x)$ if $f(x) = 2x^9 + x^e + e^x + e^3$
2. Find $f'(x)$ if $f(x) = sinxcosx$
3. Find $f'(x)$ if $f(x) = \frac{4x^6 - 1}{secx}$
4. Find $\frac{dy}{dx}$ if $2y^3 + x = \sqrt{9y + 17}$
5. Find $f^{(101)}(x)$ if $f(x) = 2e^x + 66$

Flex the Mental Muscle!

The position of a particle is given by the equation of motion $s = f(t) = 3sin(x + \frac{\pi}{2})$ (where t is in seconds and s is in meters). State the velocity and acceleration functions of the particle. At what times is the particle moving fastest? Slowest?