Topic 10 Outline

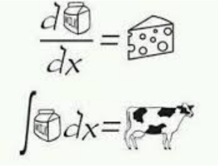
Topic 10 Learning Objectives

1. calculate antiderivatives (indefinite integrals)
2. solve basic differential equations
3. solve problems involving rectilinear motion
4. understand the area problem
5. understand and define the Reimann Sum
6. calculate basic Riemann Sums
7. understand the relationship between area and the definite integral
8. solve integrals by interpreting them as areas
9. use the Fundamental Theorem of Calculus Part 1 to calculate integrals functions
10. use the Fundamental Theorem of Calculus Part 2 to solve definite integrals
11. find areas by using definite integrals

Antiderivatives

Say we know the velocity of a particle, but want to know its position at a given time. Or, we know the rate at which bacteria is growing, but want to know the size of the popluation at a given time. In other words, given the derivative can we work *backwards* to find the original?

A function F is called an **antiderivative** of f on an interval I if $f(x) = F'(x)$ on I (ie, $f(x)$ is the derivative and $F(x)$ is the original function). The symbol that we use is called the **indefinite integral**: $\int$



Check out this link for a video on antiderivatives!
https://www.educreations.com/lesson/embed/9881168/?ref=app

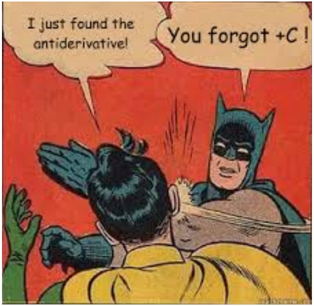
Antiderivatives

Let's examine the function $f(x) = x^2$ and see if we can come up with a conjecture for its antiderivative:

Antiderivatives

So if F and G are any 2 antiderivatives of $f(x)$, then $F'(x) = f(x) = G'(x)$ (so $F(x) - G(x) = C$, they differ only by a constant).

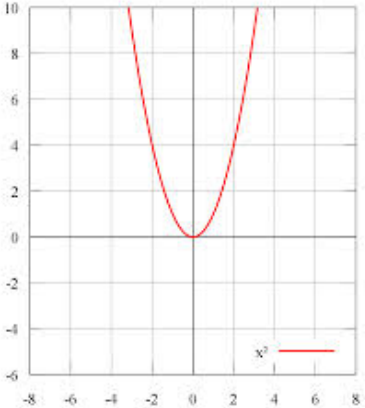
Theroem: If F is an antiderivative of f on an interval I, then the **most general antiderivative** of f on I is $F(x) + C$.



Antiderivatives

By assigning specific values to C, we obtain a family of functions whose graphs are vertical translates of one another.

Example: Sketch some members of the family of antiderivatives of $f(x) = x^2$:



Examples

State the most general antiderivatives of the following:

1. $f(x) = \sin x$

2. $f(x) = \frac{1}{x}$

3. $f(x) = x^n$

4. $f(x) = 4$

Antiderivative Formulas

Function	Antiderivative
$x^n \ (n \neq -1)$	$\frac{x^{n+1}}{n+1}$
$\frac{1}{x}$	$\ln\lvert x\rvert$
e^{ax}	$\frac{1}{a}e^{ax}$
$\sin ax$	$\frac{-1}{a}\cos ax$
$\cos x$	$\frac{1}{a}\sin x$
$\sec^2 x$	$\tan x$
$\sec x \tan x$	$\sec x$

Antiderivative Rules:

- $kf(x)$ has antiderivative $kF(x)$.
- $f(x) \pm g(x)$ has antiderivative $F(x) \pm G(x)$

Examples

Integrate:

1. $\int (3x^2 + \sec^2 x + 3)dx$
2. $\int (2e^2 t + 2e^2)dt$
3. $\int (\sqrt{x} - x^{-1})dx$
4. $\int \frac{3}{\cos^2 t} dt$
5. $\int \frac{6-x}{\sqrt[3]{x}} dx$
6. $\int x^2(x+2)dx$
7. If $f'(x) = \int (8 + 6x^2)dx$, find $f(0)$.

Differential Equations

In applications of calculus, it is very common to have a situation where it is required to find a function given knowledge about its derivative or higher-order derivatives. An equation that involves the derivatives of a function is called a **differential equation**.

Example: Find $f(x)$ if $f'(x) = e^x - x + 20$, and $f(0) = 0$

In some cases, there may be some extra conditions given that will determine the constants, and therefore *uniquely* specify the solution!

Example

Find $f(x)$ if $f''(x) = 12x^2 + 6x - 4$, and

1. $f(0) = 4$, $f(1) = 1$
2. $f'(0) = 1$, $f(0) = -3$

What do you notice from this example, in terms of solving from higher derivatives??

Rectilinear Motion

Recall that if an object has potion function $s = f(t)$, velocity is is $v(t) = s'(t)$. So position is the *antiderivative* of velocity.
Likewise, acceleration $a(t) = v'(t) = s''(t)$, so velocity is the antiderivative of accelerationMeaning that now we can move foreward or back from any of the three motion functions to find the other two!

Example

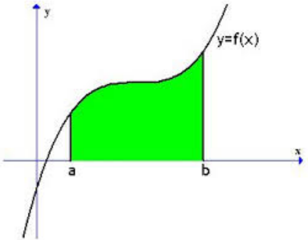
A particle moves in a straight line and has acceleration given by $a(t) = 6t + 4$. Its initial velocity is -6cm/s and initial displacement is 9cm. Find its position function $s(t)$.

The Area Problem

In this section we discover that in trying to find the area under a curve, we end up with a special type of limit.

What is the **Area Problem**??

We want to find the area of the region S that lies under the curve $y = f(x)$ $[f(x) \geq 0]$ from $x = a$ to $x = b$:



It is easy to find the area of a region with straight sides, but what can we do to estimate and eventually find the area exactly of a region with curved sides??

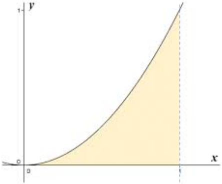
Example

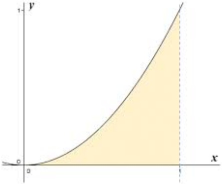
Estimate the area under the parabola $y = x^2$ (using rectangles), from $x = 0$ to $x = 1$.

How can we get a better estimate??

The Area Problem

Use more rectangles!!





If we used 1000 rectangles, $R_{1000} = 0.3338$ and $L_{1000} = 0.3328$. It seems like area $A \approx 0.333... = \frac{1}{3}$.

How many rectangles do you think we would need to take for the area to be exact??

Try it!

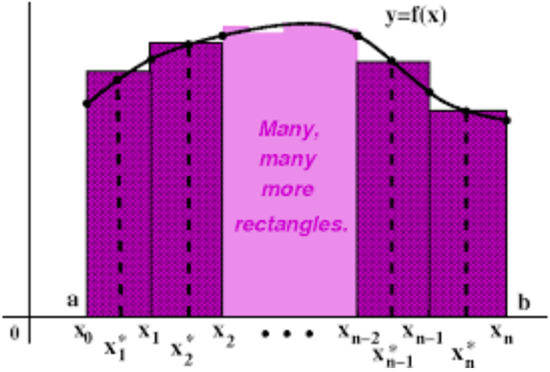
The Area Problem

Therefore we can define area to be:

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty} L_n$$

We can make our definition even more genral by not specifying where in the interval we choose to draw the rectangle from (random), and by letting the area be bounded by a general function $y = f(x)$ and the lines $x = a$ and $x = b$.

The Area Problem



Now, we approximate the area of the i^{th} strip S_i with a rectangle with width Δx and height $f(x_i^*)$, where $f(x_i^*)$ is the value of f at any number in the i^{th} subinterval $[x_{i-1}, x_i]$. We call these numbers, $x_1^*, x_2^*, \ldots.., x_n^*$ *sample points*. So:

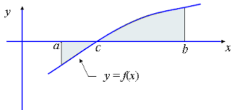
The Definite Integral

Definition: If f is a continuous function defined for $a \leq x \leq b$, we divide the interval $[a, b]$ into n subintervals of equal width $\Delta x = \frac{b-a}{n}$. Let $x_0 = a$ and $x_n - b$ be the endpoints of these subintervals, and let $x_1^*, x_2^*, \ldots.., x_n^*$ be any sample points in these subintervals, so x_i^* lies in the i^{th} subinterval $[x_{i-1}, x_i]$. Then the **definite integral** of f from $x = a$ to $x = b$ is:

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x = A$$

The Definite Integral

So far, we have restricted ourselves to the case where $f(x) \geq 0$. We can also defnite the integral is this is not so:



Example: Evaluate the following by interpreting each as an area:

1. $\int_0^3 (x-1)dx$

2. $\int_0^1 \sqrt{1-x^2}dx$

3. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin x dx$

Properties of The Definite Integral

1. $\int_a^b f(x)dx = -\int_b^a f(x)dx$
2. $\int_a^a f(x)dx = 0$
3. $\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x) \pm \int_a^b g(x)$
4. $\int_a^b c dx = c(b-a)$ (where c is a constant)
5. $\int_a^b cf(x)dx = c\int_a^b f(x)dx$
6. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$, $(a \leq c \leq b)$

Properties of The Definite Integral

1. If $f(x) \geq 0$ on $(a \leq x \leq b)$, then $\int_a^b f(x)dx \geq 0$

2. If $f(x) \leq 0$ on $(a \leq x \leq b)$, then $\int_a^b f(x)dx \leq 0$

3. If $f(x) \geq g(x)$ on $(a \leq x \leq b)$, then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$

4. If $m \leq f(x) \leq M$ on $(a \leq x \leq b)$ (with m and M beign constants), then $m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$

Examples

1. Evaluate $\int_0^1 (4+3x^2)dx$.

2. If we know that $\int_0^{10} f(x)dx = 17$ and $\int_0^8 f(x)dx = 12$, what is $\int_8^{10} f(x)dx$?

3. Estimate $\int_0^1 e^{-x^2}dx$.

The Fundamental Theorem of Calculus Part 1

We start by looking at a function defined by

$$g(x) = \int_a^x f(t)dt,$$

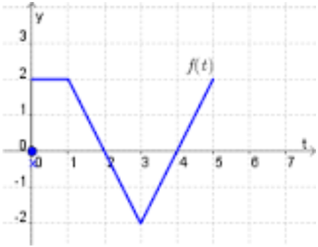
where f is a continuous function in $[a, b]$ and x varies between a and b. Thus g depends ONLY on x. If x is fixed, then $g(x) = \int_a^x f(t)dt$ is a number, but if x varies, then $g(x) = \int_a^x f(t)dt$ also varies and defines a function of x, $g(x)$!

We call this an **integral function**.

Check out this link for a video on the FTC Parts 1 and 2!
https://www.educreations.com/lesson/embed/9881561/?ref=app

Example

If f is the function whose graph is shown below, and $g(x) = \int_0^x f(t)dt$, sketch a rough graph of $g(x)$ on $[0, 5]$ by varying x.



The Fundamental Theorem of Calculus Part 1

If f is a continuous function in $[a, b]$ then the function defined by $g(x) = \int_a^x f(t)dt$, $a \leq x \leq b$ is continuous on $[a, b]$ and differentiable on (a, b), and $g'(x) = f(x)$.

If these conditions do not hold, we must modify the integral ourselves by using the properties of definite integral, or a special version of the chain rule (that we will see in the next examples).

Examples

Find the derivatives of the following:

1. $\int_0^x (4+3t^2)dt.$

2. $\int_5^x \sqrt{t+t^3}dt$

3. $\int_x^{17} \frac{e^{t^2}}{t+1}dt$

4. $\int_1^{x^4} \sec t dt$

The Fundamental Theorem of Calculus Part 2

If f is a continuous function in $[a, b]$ $\int_a^b f(x)dx = F(b) - F(a)$, where F is *any* antiderivative of f.

Evaluate the following integrals:

1. $\int_0^3 (e^x + 1)dx$.
2. $\int_{-2}^8 \sqrt[3]{x}dx$
3. $\int_0^4 (1 + 3t - t^2)dt$
4. $\int_{\pi}^{2\pi} \cos(3u)du$
5. $\int_1^2 \frac{4+x^2}{x^3}dx$

Integral Types

List some differences and identifying features of the three types of integrals that we have seen

Indefinite Integrals	Definite Integrals	Integral Functions

The Definite Integral and Area

Now that we know how to solve integrals by using antiderivatives, we can use them to solve the area problem!.

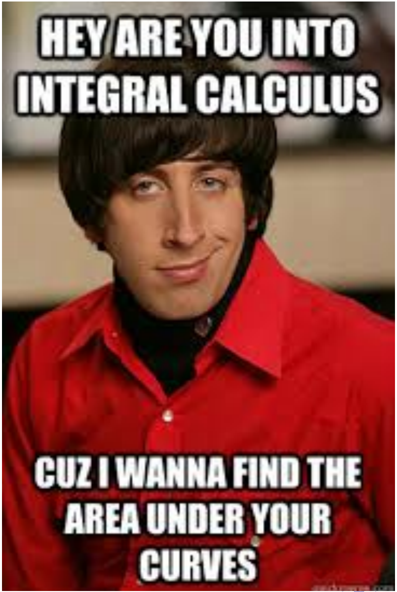
Check out this link for a video on area and the definite integral!
https://www.educreations.com/lesson/embed/9882449/?ref=app

Find the area bounded by:

1. $f(x) = x^2$, the x-axis, and the lines $x = 0$ and $x = 1$.
2. $f(x) = x - 1$, the x-axis, and the lines $x = 0$ and $x = 3$.
3. $f(x) = \sin x$, the x-axis, and the lines $x = -\frac{\pi}{2}$ and $x = \frac{\pi}{2}$.
4. the x-axis, the lines $x = -2$ and $x = 2$, and

$$f(x) = \begin{cases} x^3 & \text{if } x < 0 \\ x^2 & \text{if } x \geq 0 \end{cases}$$

The Definite Integral and Area



Five in Five!

Solve the following in 5 minutes or less!

1. Integrate $\int (x^4 - \frac{1}{\sqrt[3]{x^2}} + e^{4x} - x^{\pi} + \pi) dx$.

2. The acceleration of an object moving along the x-axis with $0 \le t \le 10$ is specified by $a(t) = 120t - 12t^2$. Furthermore, the position of the particle at $t = 0$ is 4m, and it starts from rest. State the velocity and position functions.

3. Solve $\int_0^{\pi} \sin x dx$.

4. Solve $\int_1^{e} \frac{1}{x} dx$.

5. Find the area bounded by $f(x) = |1 - x|$, the x-axis, and the lines $x = 0$ and $x = 2$.

Flex the Mental Muscle!

For the following statements, classify them as TRUE or FALSE. If they are true, give some justification, and if they are false, give a counterexample (a function, a sketch, etc) that shows that they are false.

1. We can find the antiderivative of a rational function by using the quotient rule in reverse.
2. If we wish to find a unique solution when solving for $f(t)$ from $f^{(9)}(t)$ (the 9th derivative), we must have exactly 9 conditions, one for each lower-order derivative.
3. In order to find the area under the curve $f(x)$, above the x-axis, and between the lines $x = a$ and $x = b$ using Reimann Sums, we must use infinitely many rectangles to get an exact value for area.
4. For a continuous function f(x), the definite integral of f(x) from x=a to x=b gives the area of the region bounded by f(x) and the x-axis from x=a to x=b.
5. The area of a region below the x-axis is negative.