Defn': $\lim_{x\to a} f(x) = L$ / the limit of f(x) as x approaches a, equals L

if we can make the values of f(x) arbitrarily close to L by taking x to be close to a, (on either side) $\boxed{\text{but NOT EQUAL TO } a}$.

ex//

We can find limits by constructing tables of values, or by examing graphs.

ex// The Heaviside fcn' H is defined $H(t) = \begin{cases} 0 & \text{if } t<0 \\ 1 & \text{if } t\geq 0 \end{cases}$

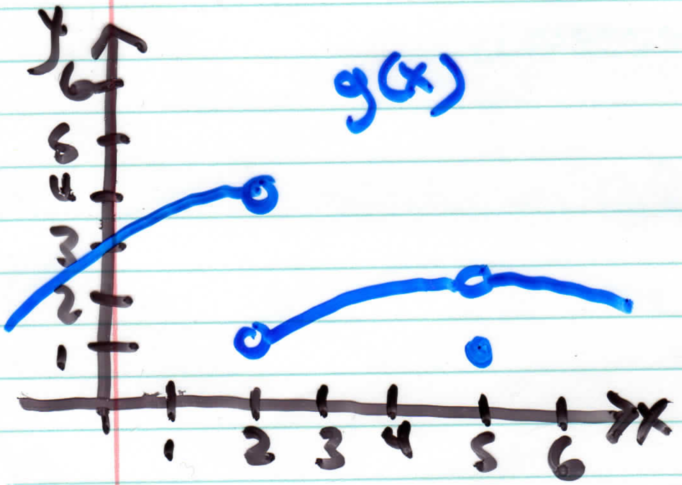
Defn: ① the limit of $f(x)$ as x approaches a from the left is written $\lim_{x \to a^-} f(x) = L$

② the limit of $f(x)$ as x approaches a from the right is written $\lim_{x \to a^+} f(x) = L$

Theorem: $\lim_{x \to a} f(x) = L$ if and only if

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L$$

ex// use the graph of g to state the values (if they exist) of the following:



a) $\lim_{x \to 2^-} g(x) =$

b) $\lim_{x \to 2^+} g(x) =$

c) $\lim_{x \to 2} g(x) =$

d) $\lim_{x \to 5^-} g(x) =$ e) $\lim_{x \to 5^+} g(x) =$ f) $\lim_{x \to 5} g(x) =$ e) $f(5) =$

ex// Sketch the graph of the following fcn' & use it to determine the values of a for which $\lim_{x \to a} f(x)$ exists: $f(x) = \begin{cases} 2-x & \text{if } x < -1 \\ x & \text{if } -1 \leq x < 1 \\ (x-1)^2 & \text{if } x \geq 1 \end{cases}$

Infinite Limits: For some fcn's, the values of $f(x)$ get very large (large negatively) at some values of x.

ex// find $\lim_{x \to 0} \frac{1}{x^2}$ (if it exists)

Defn': Let f be a fcn' defined on both sides of a, except at a itself, then $\lim_{x \to a} f(x) = \infty \ (-\infty)$ means that the values of f(x) can be made arbitrarily large (large negatively) by taking x close to a, but not equal to a.

ex// determine the infinite limits:

a) $\lim_{x \to 5} \frac{6}{x-5}$

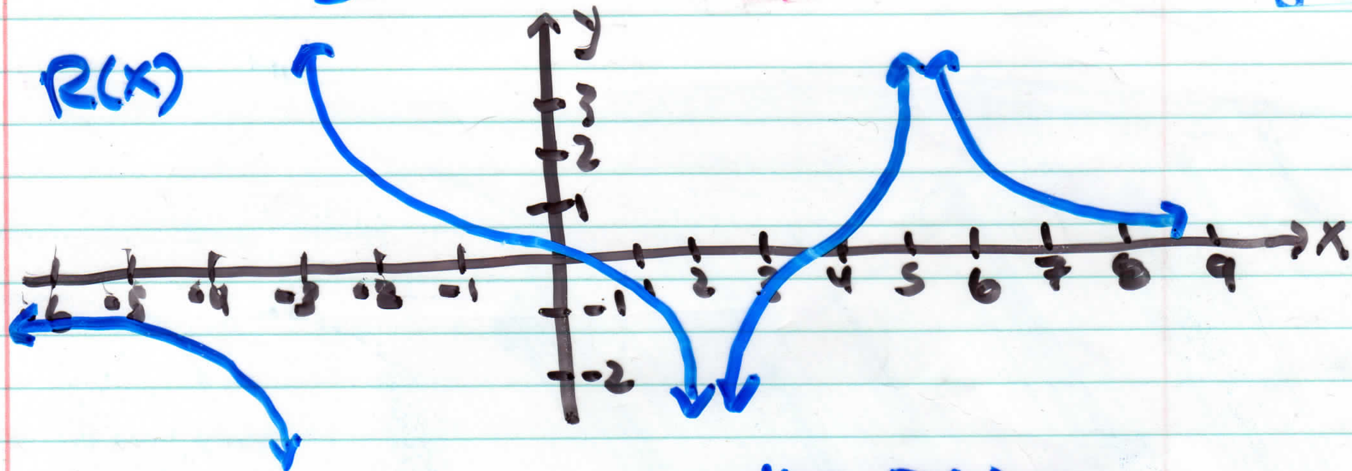
b) $\lim_{x \to 1} \frac{2-x}{(x-1)^2}$

Defn': the line $x=a$ is called a "vertical asymptote" (v.a.) if at least ONE of the following is true:

$\lim_{x\to a} f(x) = \infty / -\infty$, $\lim_{x\to a^-} f(x) = \infty / -\infty$, $\lim_{x\to a^+} f(x) = \infty / -\infty$

ex// For R(x) given below, state the following:

R(x)



a) $\lim_{x\to 2} R(x) =$

b) $\lim_{x\to 5} R(x) =$

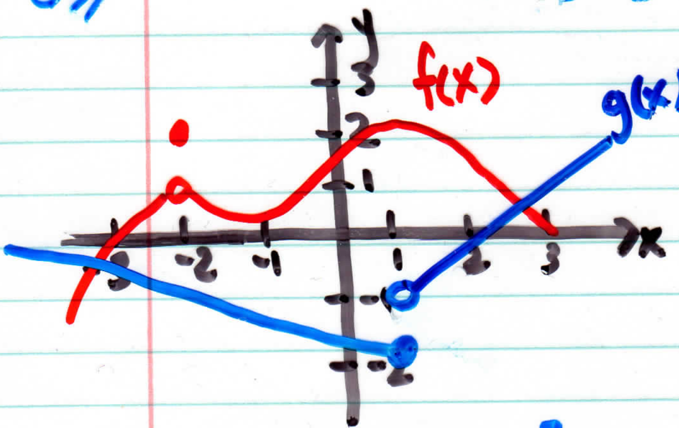
c) $\lim_{x\to -3} R(x) =$

d) the equations of any vertical asymptotes.

Section 2.3: Calculating Limits Using Limit Laws

In the last section we used tables of values & graphs to guess limits, we can calculate exact limits by using the Limit Laws

ex// Use the limit Laws & the graph to evaluate:



a) $\lim_{x \to -2} [f(x) + 5g(x)] =$

b) $\lim_{x \to 1} [f(x)g(x)] =$

c) $\lim_{x \to 2} \frac{f(x)}{g(x)} =$

d) $\lim_{x \to 1} [f(x)]^3 =$

e) $\lim_{x \to -2} \sqrt[10]{f(x)} =$

Limit Laws: Suppose c is a constant and $\lim_{x\to a} f(x) = A$ & $\lim_{x\to a} g(x) = B$ (ie, both exist), then:

1. $\lim_{x\to a} [f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) = A+B$
2. $\lim_{x\to a} [f(x)-g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x) = A-B$
3. $\lim_{x\to a} [cf(x)] = c\lim_{x\to a} f(x) = cA$
4. $\lim_{x\to a} [f(x)g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x) = A \cdot B$
5. $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{A}{B}$
6. $\lim_{x\to a} [f(x)]^n = [\lim_{x\to a} f(x)]^n = A^n$
7. $\lim_{x\to a} c = c$
8. $\lim_{x\to a} x = a$
9. $\lim_{x\to a} x^n = a^n$ &
10. $\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a} = a^{1/n}$ (for n being a positive integer)
11. $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)} = \sqrt[n]{A} = A^{1/n}$

Direct Substitution Property: If f is a polynomial or rational fcn' & a is in the domain of f, then:

$$\lim_{x\to a} f(x) = f(a)$$

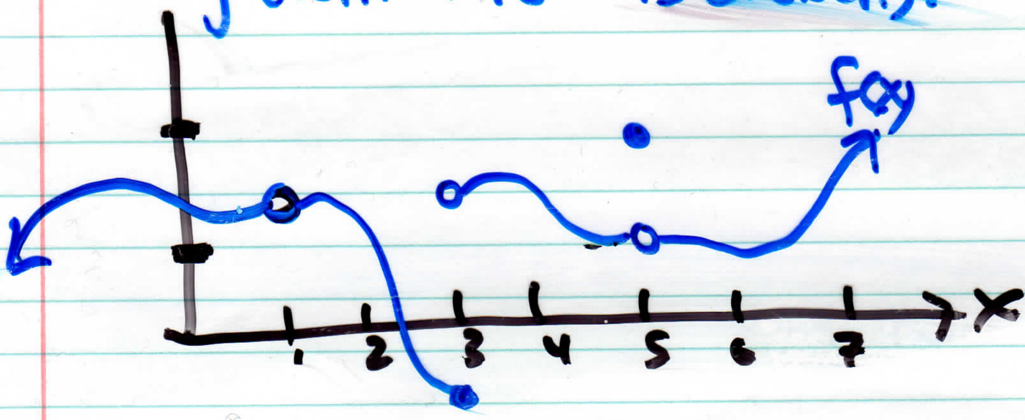
Section 2.5: Continuity

We noticed in the last section that $\lim_{x \to a} f(x)$ can often be found just by plugging in a (ie, $f(a)$). Functions with this property are called "continuous at a" ie, when $\lim_{x \to a} f(x) = f(a)$ (DSP)

Otherwise, we say $f(x)$ is "discontinuous at a"

Geometrically, a continuous (cts) fcn' has a graph with no breaks in it.

ex// At which numbers is f discontinuous? Why (which rule is broken)?



Theorem ①: If f & g are continuous at a & c is a constant, then the following fcn's are also cts at a.

① $f+g$ ② $f-g$

③ cf ④ fg

⑤ $\frac{f}{g}$ (if $g(a)\neq 0$) ⑥ $f\circ g = f(g(x))$ (if f is cts. at $g(a)$).

Theorem ②: The following types of fcn's are cts. at every number in their domains:

polynomial rational fcn's root fcn's
trig fcn's inverse trig fcn's
exponential fcn's *logarithmic functions*

ex/ Prove ① from Theorem ①

Intermediate Value Theorem: Suppose f is cts. on the closed interval $[a,b]$ & let N be any number between $f(a)$ & $f(b)$ (where $f(a) \neq f(b)$.) Then there exists a number c in (a,b) such that $f(c) = N$

ex// Show that there is a of the eqn' $4x^3 - 6x^2 + 3x - 2 = 0$ between 1 & 2.

ex// Prove that $e^x = 2 - x$ has at least one real root.

Section 2.6: Limits at Infinity (Horizontal Asymptotes)

We already know what a vertical asymptote is & how to find one

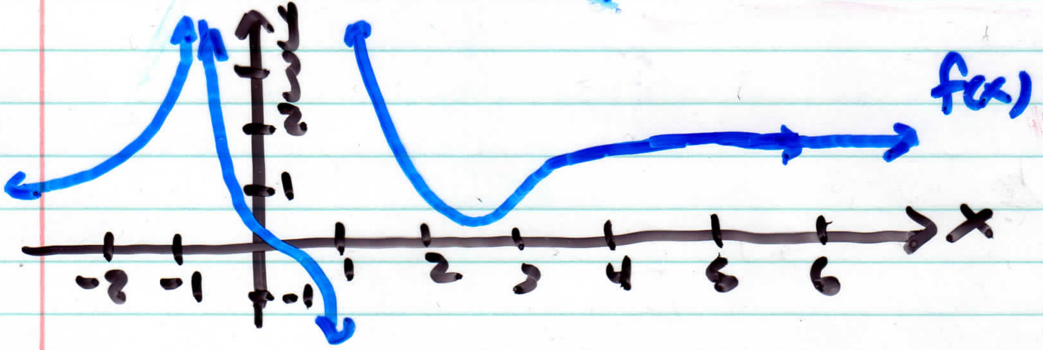
Now, we let x become larger & larger (positive or negative) & see what happens to y.

For example, what happens to $\frac{1}{x}$ as $x \to \pm\infty$?

In this case, $y=0$ is called a "horizontal asymptote"

Defn': The line $y=L$ is called a "horizontal asymptote" (h.a.) if either $\lim_{x\to\infty} f(x)=L$ or $\lim_{x\to-\infty} f(x)=L$.

ex/ Find the infinite limits (v.a.'s) & limits at infinity (h.a.'s)

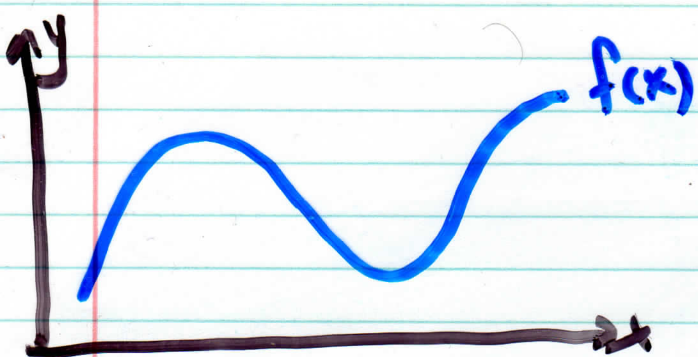


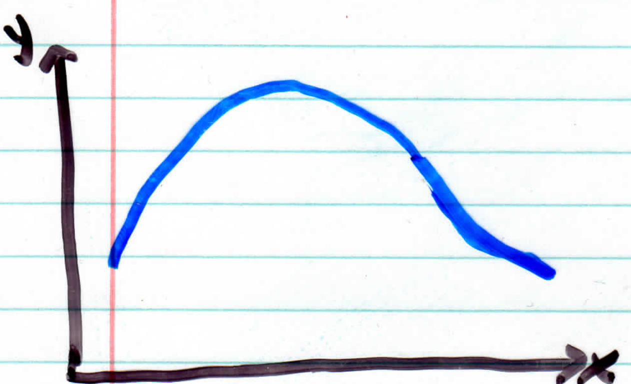
Theorem': If $r>0$ is a rational number, then $\lim_{x\to\infty} \frac{1}{x^r}=0$.

If $r>0$ is a rational number such that x^r is defined for all x, then $\lim_{x\to-\infty} \frac{1}{x^r}=0$.

Section 2.7: Tangents, Velocities, & other Rates of Change

A "tangent" to a curve at a point is a line that touches the curve at that point





Rates of Change: If x changes from x_1 to x_2 & $y = f(x)$, then the change in x is $\Delta x = x_2 - x_1$ & the corresponding change in y is $\Delta y = f(x_2) - f(x_1)$.

The Average Rate of Change: $\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$ (slope of secant line)

Instantaneous Rate of Change: $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1}$

One such rate of change is velocity. If an object moves along a straight line according to an eq'n of motion $s = f(t)$, where s is the displacement of the object at time t. Then f is called the position function. From time $t = a$ to time $t = a + h$, the

average velocity $= \frac{\text{displacement}}{\text{time}} = \frac{f(a+h) - f(a)}{h}$

As the time intervals get shorter (ie, we let $h \to 0$), then the Instantaneous Velocity is:

$$\text{Velocity} = v(a) = \boxed{\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}}$$

ex// Suppose a ball is dropped from the upper observation deck of the CN Tower, 450 m high, & the distance (in meters) fallen after t seconds is $4.9t^2$.

a) What is the balls velocity after 5 sec?

b) How fast is the ball travelling when it hits the ground?

Section 2.7 Derivatives

Recall: The slope of the tangent line to the curve $y=f(x)$ at the point where $x=a$ $[(a, f(a))]$ is:

$$m = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$$

x y

& the velocity of an object with position function $s=f(t)$ at time $t=a$ is:

$$v(a) = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$$

In fact, this type of limit occurs widely in any of the sciences or engineering whenever we calculate a rate of change, so we give it a special name & notation:

Section 2.9: The Derivative as a Function.

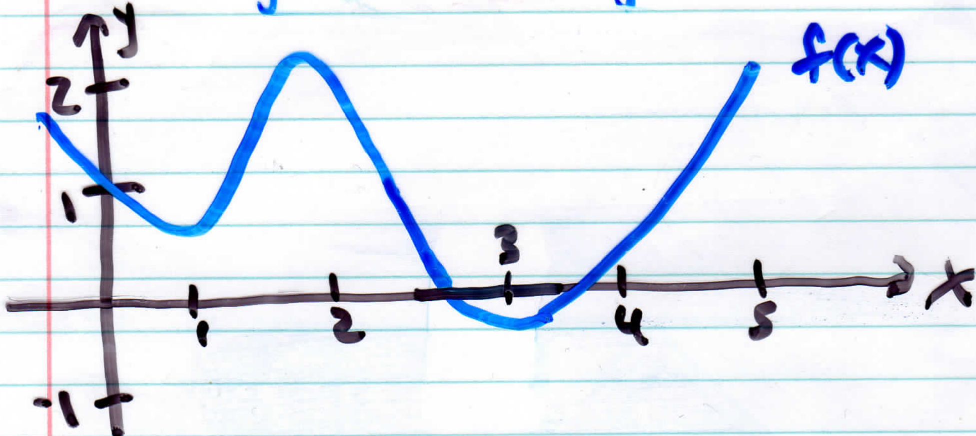
Recall: We defined the derivative of a function f at a **fixed** number a: $f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$

If we replace a in the above eqn' by the **variable** x, then we can vary a (choose different a values).

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Now the derivative $f'(x)$ is a new function of x.

ex/ Use the graph of f(x) given to sketch the graph of $f'(x)$



Some other notations we use for derivatives of $y=f(x)$:

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x) = D\,f(x) = D_x f(x)$$

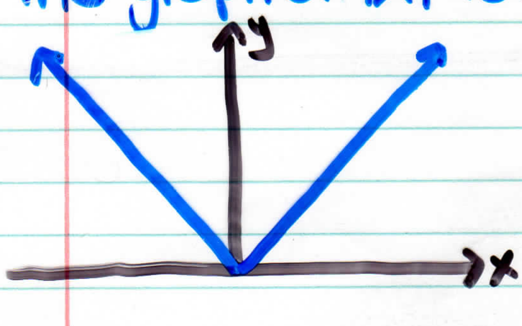
or $f'(a) = \left.\frac{dy}{dx}\right|_{x=a}$

<u>Defn</u>: A fcn f is "differentiable at a" if $f'(a)$ exists. It is differentiable on an open interval (a,b) [or (a,∞), $(-\infty,a)$, $(-\infty,\infty)$] if it is differentiable at every number in the interval.

ex) In our previous example, where is f differentiable?

In general, if the graph of a function $f(x)$ has a "corner" or "kink" then the graph has no tangent at this point.

ex) The graph of $|x|$ is:



<u>Theorem</u>: If f is differentiable at a, then f is cts. at a (so if f is discts., it is NOT differentiable (at that point) BUT just because f is cts. does NOT make it differentiable there ie, $|x|$).