[9] 1. Consider the linear system:

$$\begin{array}{rcrcrcrcl} x_1 & + & 3x_2 & & & + & 6x_4 & = & 4 \\ x_1 & + & 3x_2 & + & 4x_3 & - & 2x_4 & = & 8 \end{array}$$

RREF

(a) Find the general solution to this system using Gauss-Jordan elimination.

$$\left[\begin{array}{cccc|c} 1 & 3 & 0 & 6 & 4 \\ 1 & 3 & 4 & -2 & 8 \end{array}\right] R_2 \to R_2 - R_1 \left[\begin{array}{cccc|c} 1 & 3 & 0 & 6 & 4 \\ 0 & 0 & 4 & -8 & 4 \end{array}\right] R_2 \to R_2/4$$

$$\left[\begin{array}{cccc|c} 1 & 3 & 0 & 6 & 4 \\ 0 & 0 & 1 & -2 & 1 \end{array}\right] \Rightarrow$$

free free

$x_4 = t$
$x_2 = s$
$x_3 = 1 + 2t$
$x_1 = 4 - 3s - 6t$

(b) Find a solution to the above system with $x_2 = -2$ and $x_4 = 3$.

$x_1 = 4 - 3(-2) - 6(3) = 4 + 6 - 18 = \underline{\underline{-8}}$
$x_2 = -2$ //
$x_3 = 1 + 2(3) = 7$ //
$x_4 = 3$ //

[8] 2. Let $A = \begin{bmatrix} 3 & 2 \\ -4 & 8 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 2 & -1 \\ 2 & 1 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 3 & 2 \\ -2 & 0 \\ 0 & 4 \end{bmatrix}$.

In each part below, evaluate the expression or state that it does not exist. If the expression does not exist, give a reason.

(a) $AB + C$

$(2\times2)(2\times3)$
$(2\times3) + (3\times2)$ → Does not exist because AB is not the same size as C

(b) $AC + B$

$(2\times2)(3\times2)$ → Does not exist because the number of columns in A doesn't match number of ro[ws] in C (so AC is undefined)

(c) $BC + A$

$(2\times3)(3\times2)$
$(2\times2) + (2\times2)$

$BC = \begin{bmatrix} 5 & 2 \\ 4 & 8 \end{bmatrix}$, $BC + A = \begin{bmatrix} 8 & 4 \\ 0 & 16 \end{bmatrix}$ //

[9] 3. Let $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & -1 & 0 \end{bmatrix}$. Find A^{-1} by the method of row reduction. Show all your work. Write your final answer where indicated at the bottom of the page. (*No Credit* for any other method.)

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 2 & -1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 2 & 1 & 1 & 1 & 0 & 0 \\ 2 & -1 & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} R_2 \to R_2 - 2R \\ R_3 \to R_3 - 2R \end{array}$$

$$A \qquad I$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 & -2 & 0 \\ 0 & -1 & -2 & 0 & -2 & 1 \end{array}\right] \xrightarrow{R_3 \to R_3 + R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 & -2 & 0 \\ 0 & 0 & -3 & 1 & -4 & 1 \end{array}\right] R_3 \to R3/-3$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 & -2 & 0 \\ 0 & 0 & 1 & -\frac{1}{3} & \frac{4}{3} & -\frac{1}{3} \end{array}\right] \begin{array}{l} R_1 \to R_1 - R_3 \\ R_2 \to R_2 + R_3 \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ 0 & 1 & 0 & \frac{2}{3} & -\frac{2}{3} & -\frac{1}{3} \\ 0 & 0 & 1 & -\frac{1}{3} & \frac{4}{3} & -\frac{1}{3} \end{array}\right] \to A^{-1}!$$

check:
$$\underset{A}{\begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & -1 & 0 \end{bmatrix}} \underset{A^{-1}}{\begin{bmatrix} \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{4}{3} & -\frac{1}{3} \end{bmatrix}} = \underset{I}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \checkmark$$

Answer: $A^{-1} = \begin{bmatrix} \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{4}{3} & -\frac{1}{3} \end{bmatrix}$

[9] 4. Express $A = \begin{bmatrix} 0 & 3 \\ 1 & 5 \end{bmatrix}$ as a product of elementary matrices. Show all your work.

Step 1: Get $A \to I$, $\begin{bmatrix} 0 & 3 \\ 1 & 5 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 5 \\ 0 & 3 \end{bmatrix} \xrightarrow{R_2 \to R_2/3} \begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix} \xrightarrow{R_1 \to R_1 - 5R_2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$E_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $E_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1/3 \end{bmatrix}$ $E_3 = \begin{bmatrix} 1 & -5 \\ 0 & 1 \end{bmatrix}$

2 $E_3 E_2 E_1 A = I \Rightarrow A = E_1^{-1} E_2^{-1} E_3^{-1} I = E_1^{-1} E_2^{-1} E_3^{-1}$

3 $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ 1 & 5 \end{bmatrix}$ ✓

[9] 5. Let $A = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -4 & 1 & 2 \end{bmatrix}$, and assume B is another 3×3 matrix with $\det(B) = 5$.

(a) Find $\det(A)$, by expansion along row 2. (*No Credit* for any other method.)

$\det(A) = -0 \begin{vmatrix} 2 & 2 \\ 1 & 2 \end{vmatrix} + 2 \begin{vmatrix} 1 & 2 \\ -4 & 2 \end{vmatrix} - 1 \begin{vmatrix} 1 & 2 \\ -4 & 1 \end{vmatrix}$

$= 0 + 2(2-(-8)) - 1(1-(-8)) = 2(10) - 1(9) = 11$ //

(b) Find the determinant of BAB^T.

$\det(BAB^T) = \det(B)\det(A)\det(B^T)$

$= \det(B)\det(A)\det(B) = 5(11)(5) = 275$ //

(c) Find the determinant of $(2B)A^{-1}$.

$\det((2B)A^{-1}) = \det(2B)\det(A^{-1})$

$= 2^3 \det(B)\det(A^{-1}) = 8(5)\left(\frac{1}{11}\right) = \frac{40}{11}$ //

[8] 6. Use Cramer's rule to solve the following system. (*No Credit* for any other method)

$$4x - 2y = 4$$
$$3x + y = 3$$

$A = \begin{bmatrix} 4 & -2 \\ 3 & 1 \end{bmatrix}$, $b = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$ $\longrightarrow$ $\det(A) = 4 + 6 = 10$

$A_1 = \begin{bmatrix} 4 & -2 \\ 3 & 1 \end{bmatrix}$, $\det(A_1) = 10$, $A_2 = \begin{bmatrix} 4 & 4 \\ 3 & 3 \end{bmatrix}$, $\det(A_2) = 12 - 12 = 0$

So $x_1 = \dfrac{\det(A_1)}{\det(A)} = \dfrac{10}{10} = 1$ & $y = \dfrac{\det(A_2)}{\det(A)} = \dfrac{0}{10} = 0$

[8] 7. Assume that the augmented matrix of a certain linear system can be reduced to

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 4 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & a & b \end{array}\right]$$

with elementary row operations.

Determine all values of a and b (if any) for which this system

(a) has **no** solutions:

$a = 0$, $b \neq 0$ (then it will be inconsistent)

(b) has a **unique** solution:

$a \neq 0$, $b \neq 0$ (then $x_3 = b/a$ & solve for x_1 & x_2)

(c) has **infinitely many** solutions:

$a = 0$, $b = 0$ (then x_3 will be "free")

(d) In case (c), determine the **general solution.**

$x_3 = t$, $x_2 = 5 - 2t$, $x_1 = 4 - 3t$