CHAPTER 3: Vectors in 2-space & 3-Space

Many physical quantities can be completely described by their size ("magnitude"):

These are called "scalars".

Other physical quantities also need to be described in terms of their direction:

These are called "vectors".

Section 3.1: Introduction to Vectors (Geometric)

Vectors can be represented in 2-D or 3-D as directed line segments (arrows).

Vectors with the same length & direction are called "equivalent" (even if they are located in different positions.

Defn': The SUM $\vec{v}+\vec{w}$ is the vector determined as follows: Position $\vec{w}$ so that its initial point coincides with the terminal point of $\vec{v}$.

Defn': A vector of length 0 is the "zero vector," $\vec{0}$.

$$\vec{0}+\vec{v}=\vec{v}+\vec{0}=\vec{v}$$

Defn': The NEGATIVE of $\vec{v}$, $-\vec{v}$, is the vector that has the same magnitude as $\vec{v}$ but is oppositely directed.

Defn': The DIFFERENCE is $\vec{v}-\vec{w}=\vec{v}+(-\vec{w})$.

__Defn'__: If $\vec{v}$ is a non-zero vector & k is a non-zero real number (scalar), then the **PRODUCT** $k\vec{v}$ is the vector whose length is $|k|$ times the length of $\vec{v}$ & whose direction is the same as $\vec{v}$ if $k > 0$ & opposite to $\vec{v}$ if $k < 0$.

A great way to describe/draw/analyze vectors is within a coordinate system.

Let's start with the plane/2-space.

If we position $\vec{v}$ at the origin, then (v_1, v_2) are *"components"* of $\vec{v}$:

Vectors in 3-Space (3D):

We have similar results for vectors in 3-space as we did for vectors in 2-space

1) $\vec{v} = \vec{w}$ if and only if $v_1 = w_1$, $v_2 = w_2$, $v_3 = w_3$
2) $\vec{v} \pm \vec{w} = (v_1 \pm w_1, v_2 \pm w_2, v_3 \pm w_3)$
3) $k\vec{v} = (kv_1, kv_2, kv_3)$

ex/ For $\vec{v} = (-1, 0, 3)$ & $\vec{w} = (2, 3, 1)$, draw $\vec{v}$ & $\vec{w}$, find $2\vec{v}$, $\vec{v} + \vec{w}$, $2\vec{v} - \vec{w}$

If a vector is positioned so that its initial point is not at the origin, then:

ex// What are the components of the vector $\vec{v}$ with initial point $(10, -7, 3)$ & terminal point $(11, 3, -2)$?

Sometimes we can simplify a problem by "translating" the coordinate axis:

ex// Suppose we translate to obtain an x'-y' system whose origin is $(k, l) = (4, 1)$

a) find x', y' coords. of the point $(x, y) = P(2, 0)$

b) find x, y coords. of the point $(x', y') = Q(-1, 5)$

Section 3.2: Norm of a Vector; Vector Arithmetic

Properties of Vector Arithmetic:

If $\vec{u}$ & $\vec{v}$ are vectors in 2- or 3-space & k & l are scalars, then:

1) $\vec{u}+\vec{v}=\vec{v}+\vec{u}$
2) $(\vec{u}+\vec{v})+\vec{w}=\vec{u}+(\vec{v}+\vec{w})$
3) $\vec{u}+\vec{0}=\vec{0}+\vec{u}=\vec{u}$
4) $\vec{u}+(-\vec{u})=0$
5) $k(l\vec{u})=(kl)\vec{u}$
6) $k(\vec{u}+\vec{v})=k\vec{u}+k\vec{v}$
7) $(k+l)\vec{u}=k\vec{u}+l\vec{u}$
8) $1\vec{u}=\vec{u}$

Defn: The length of a vector is often called its "norm", denoted $\|\vec{u}\|$: $\|\vec{u}\|=\sqrt{u_1^2+u_2^2+u_3^2}$

__Note__: A vector of norm (length) 1 is a

If P_1 & P_2 are 2 points in 2- or 3- space, the distance between them is...

ex// For $\vec{u} = (-3, 2, \sqrt{3})$, what is:

a) $\|\vec{u}\|$

b) $\frac{1}{\|\vec{u}\|}\vec{u}$

c) $\left\|\frac{1}{\|\vec{u}\|}\vec{u}\right\|$

Section 3.3: Dot Product; Projections

If we position $\vec{u}$ & $\vec{v}$ so that their initial point coincide, we can define the "angle between $\vec{u}$ & $\vec{v}$," θ, where $0 \le \theta \le \pi$;

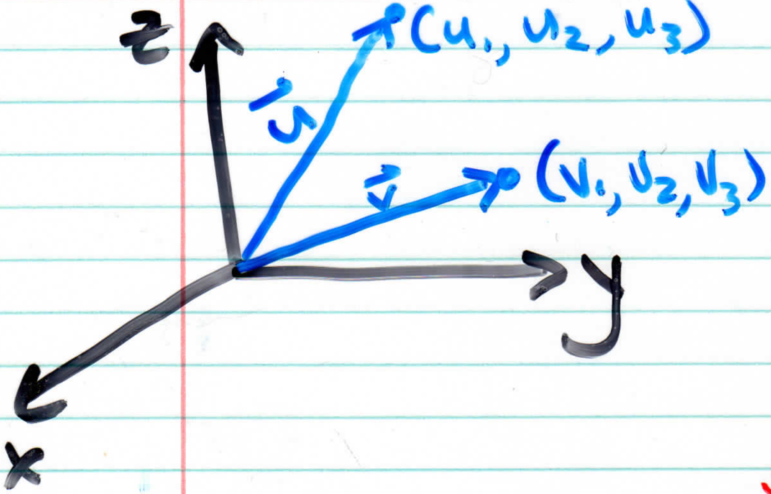
Defn: If $\vec{u}$ & $\vec{v}$ are vectors in 2- or 3-space & θ is the angle between $\vec{u}$ & $\vec{v}$, then the "Euclidean inner product" or "dot product", $\vec{u} \cdot \vec{v}$, is:

$$\vec{u} \cdot \vec{v} = \begin{cases} \|\vec{u}\| \, \|\vec{v}\| \cos\theta & \text{if } \vec{u} \ne 0 \text{ \& } \vec{v} \ne 0 \\ 0 & \text{if } \vec{u} = 0 \text{ or } \vec{v} = 0 \end{cases}$$

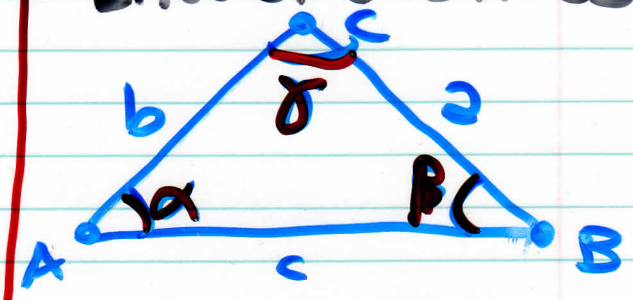
ex// For $\vec{u} = (0, -1, 0)$ & $\vec{v} = (0, 2, 0)$, find $\vec{u} \cdot \vec{v}$.

We can also find a formula for the dot product in terms of components:

Let $\vec{u} = (u_1, u_2, u_3)$ & $\vec{v} = (v_1, v_2, v_3)$ $(\neq 0)$,



LAW OF COSINES



$$c^2 = a^2 + b^2 - 2ab\cos(\gamma)$$

or sometimes, to find θ:

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \, \|\vec{v}\|}$$

ex// For $\vec{u} = (2,-1,1)$ & $\vec{v} = (1,1,2)$ find $\vec{u}\cdot\vec{v}$ & determine the angle between $\vec{u}$ & $\vec{v}$.

<u>Thm</u>: If $\vec{u}$ & $\vec{v}$ are vectors in 2- or 3-space:

a) $\vec{v}\cdot\vec{v} = \|\vec{v}\|^2$, ie, $\|\vec{v}\| = (\vec{v}\cdot\vec{v})^{1/2}$

b) If $\vec{u}$ & $\vec{v}$ are non-zero, & θ is the angle between them:

i) θ is acute $(0<\theta<\pi/2)$ if & only if $\vec{u}\cdot\vec{v}>0$

ii) θ is obtuse $(\pi/2<\theta<\pi)$ " $\vec{u}\cdot\vec{v}<0$

iii) $\theta = \pi/2$ if and only if $\vec{u}\cdot\vec{v}=0$

ex// For $\vec{u}=(1,5,-2)$, $\vec{v}=(3,-1,-1)$, $\vec{w}=(2,3,4)$, Find $\vec{u}\cdot\vec{v}$, $\vec{u}\cdot\vec{w}$, $\vec{v}\cdot\vec{w}$ and describe the angles between those vectors:

If the angle between two vectors is $\pi/2$, they are perpendicular, or "orthogonal", (ie, $\vec{u} \perp \vec{v}$). In other words, 2 vectors are orthogonal if & only if their dot product is 0 (note: if either or both are $=\vec{0}$, we consider them perpendicular).

ex// Show that in 2-space, the non-zero vector $\vec{n}=(a,b)$ is $\perp$ to the line $ax+by=c$.

Properties of the Dot Product:

If $\vec{u}$, $\vec{v}$ & $\vec{w}$ are vectors in 2- or 3-space, & k is a scalar, then

a) $\vec{u}\cdot\vec{v} = \vec{v}\cdot\vec{u}$ b) $\vec{u}\cdot(\vec{v}+\vec{w}) = \vec{u}\cdot\vec{v} + \vec{u}\cdot\vec{w}$

c) $k(\vec{u}\cdot\vec{v}) = (k\vec{u})\cdot\vec{v} = \vec{u}\cdot(k\vec{v})$

d) $\vec{v}\cdot\vec{v} > 0$ if $\vec{v} \neq 0$, & $\vec{v}\cdot\vec{v} = 0$ if $\vec{v} = 0$

Sometimes we want to decompose a vector $\vec{u}$ into a sum of two others, based on some third vector $\vec{a}$. One of these will be parallel to $\vec{a}$, the other will be $\perp$ to $\vec{a}$.

We do this by forming the difference $\vec{w}_2 = \vec{u} - \vec{w}_1$

$\vec{w}_1$: orthogonal projection of $\vec{u}$ on $\vec{a}$ $= \text{proj}_{\vec{a}}\vec{u}$
vector component of $\vec{u}$ along $\vec{a}$

$\vec{w}_2$: vector component of $\vec{u}$ orthogonal to $\vec{a}$

$= \vec{u} - \text{proj}_{\vec{a}}\vec{u}$

Thm: If $\vec{u}$ & $\vec{a}$ are vectors in 2- or 3-space, & if $\vec{a} \neq 0$, then:

Note 1: If $\vec{u} \perp \vec{a}$; $\text{proj}_{\vec{a}} \vec{u} = 0$

$\vec{u} - \text{proj}_{\vec{a}} \vec{u} = \vec{u}$

If $\vec{u}$ parallel to $\vec{a}$;

$$\text{proj}_{\vec{a}} \vec{u} = \begin{cases} \dfrac{\|\vec{u}\| \, \|\vec{a}\| \cos 0}{\|\vec{a}\|^2} \vec{a} \\ \dfrac{\|\vec{u}\| \, \|\vec{a}\| \cos \pi}{\|\vec{a}\|^2} \vec{a} \end{cases}$$

Note 2: $\text{proj}_{\vec{a}} \vec{u} \perp \vec{u} - \text{proj}_{\vec{a}} \vec{u}$

ex// Let $\vec{u} = (2,-6,5)$, Find the vector component of $\vec{u}$ along $\vec{a}$ & the vector component of $\vec{u}$ orthogonal to $\vec{a}$ if:

a) $\vec{a} = (1,-3,0)$

b) $\vec{a} = (10, 0, -4)$

c) $\vec{a} = (4, -12, 10)$

Note 3: $\|\text{proj}_{\vec{a}}\, \vec{u}\| =$

We can use this knowledge to find a formula for the distance between a point $P_0(x_0, y_0)$ & a line $ax+by=c$.

Let $Q(x_1, y_1)$ be a point on the line, & pick $\vec{n} = (a, b)$

ex// What is the distance from the point $(-2,5)$ to the line $2x-y=3$?

Section 3.4: The Cross Product

We defined (in 3.3) a product of vectors that results in a scalar — the *dot product*. In this section we will define a type of vector multiplication that results in a vector (but it is only applicable in 3-space).

Defn': If $\vec{u}=(u_1,u_2,u_3)$ & $\vec{v}=(v_1,v_2,v_3)$ are vectors in 3-space, then the "cross-product" $\vec{u}\times\vec{v}$ is the vector defined by:

$$\vec{u}\times\vec{v}=(u_2v_3-u_3v_2,\ u_3v_1-u_1v_3,\ u_1v_2-u_2v_1),\ \text{or}\ldots$$

Consider the vectors $\vec{i}=(1,0,0)$, $\vec{j}=(0,1,0)$, $\vec{k}=(0,0,1)$:

Note: $\vec{i}\times\vec{i}=\vec{0}$ $\quad \vec{j}\times\vec{j}=0$ $\quad \vec{k}\times\vec{k}=0$

$\vec{i}\times\vec{j}=\vec{k}$ $\quad \vec{j}\times\vec{k}=\vec{i}$ $\quad \vec{k}\times\vec{i}=\vec{j}$

$\vec{j}\times\vec{i}=-\vec{k}$ $\quad \vec{k}\times\vec{j}=\vec{i}$ $\quad \vec{i}\times\vec{k}=\vec{j}$

ex// find $\vec{u}\times\vec{v}$ where $\vec{u}=(-3,0,2)$ & $\vec{v}=(1,-1,5)$

Cross- & Dot- Product Relationships

a) $\vec{u}\cdot(\vec{u}\times\vec{v})=0$

b) $\vec{v}\cdot(\vec{u}\times\vec{v})=0$

c) $\|\vec{u}\times\vec{v}\|^2=\|\vec{u}\|^2\|\vec{v}\|^2-(\vec{u}\cdot\vec{v})^2$

d) $\vec{u}\times(\vec{v}\times\vec{w})=(\vec{u}\cdot\vec{w})\vec{v}-(\vec{u}\cdot\vec{v})\vec{w}$

e) $(\vec{u}\times\vec{v})\times\vec{w}=(\vec{u}\cdot\vec{w})\vec{v}-(\vec{v}\cdot\vec{w})\vec{u}$

ex// Consider the vectors $\vec{u}=(-3,0,2)$ & $\vec{v}=(1,-1,5)$, calculate $\vec{u}\cdot(\vec{u}\times\vec{v})$ & $\vec{v}\cdot(\vec{u}\times\vec{v})$

Properties of the Cross-Product:

If $\vec{u}$, $\vec{v}$ & $\vec{w}$ are any vectors in 3-space, & k is any scalar, then;

a) $\vec{u}\times\vec{v}=-(\vec{v}\times\vec{u})$

b) $\vec{u}\times(\vec{v}+\vec{w})=(\vec{u}\times\vec{v})+(\vec{u}\times\vec{w})$

c) $(\vec{u}+\vec{v})\times\vec{w}=(\vec{u}\times\vec{w})+(\vec{v}\times\vec{w})$

d) $k(\vec{u}\times\vec{v})=(k\vec{u})\times\vec{v}=\vec{u}\times(k\vec{v})$

e) $\vec{u}\times\vec{0}=\vec{0}\times\vec{u}=\vec{0}$

f) $\vec{u}\times\vec{u}=\vec{0}$

We have a nice, geometric representation for the cross product via Lagrange's Identity:

Thm: If $\vec{u}$ & $\vec{v}$ are vectors in 3-space, then $\|\vec{u}\times\vec{v}\|$ is equal to the area of the parallelogram determined by $\vec{u}$ & $\vec{v}$.

ex// Find the area of the parallelogram determined by $\vec{u}=(1,2,-1)$ & $\vec{v}=(4,0,-2)$

ex// Find the area of the triangle determined by the points $P_1(2,2,0)$, $P_2(-1,0,2)$, & $P_3(0,4,3)$

Defn: If $\vec{u}$, $\vec{v}$, & $\vec{w}$ are vectors in 3-space, then $\vec{u}\cdot(\vec{v}\times\vec{w})$ is the "scalar triple product" of $\vec{u}$, $\vec{v}$ & $\vec{w}$:

$$\vec{u}\cdot(\vec{v}\times\vec{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

ex// Calculate $\vec{u}\cdot(\vec{v}\times\vec{w})$ of the vectors $\vec{u} = 3\vec{i} - 2\vec{j} - 5\vec{k}$, $\vec{v} = \vec{i} + 4\vec{j} - 4\vec{k}$, $\vec{w} = 3\vec{j} + 2\vec{k}$

If we interchange 2 rows at a time:

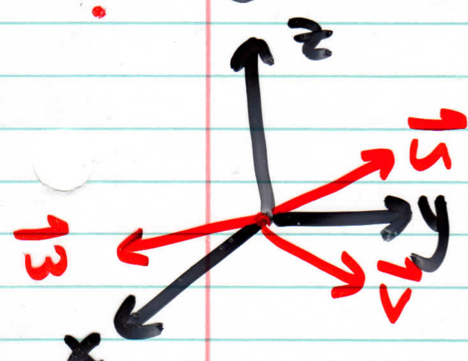
$\vec{u}\cdot(\vec{v}\times\vec{w})=\vec{w}\cdot(\vec{u}\times\vec{v})=\vec{v}\cdot(\vec{w}\times\vec{u})$

Now we can have a geometric interpretation for determinants!

<u>Thm'</u>: a) $\left|\det\begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix}\right|$ = area of par. (2-space) determined by $\vec{u}$ & $\vec{v}$

b) $\left|\det\begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix}\right|$ = volume of "parallelopiped" (3-space) determined by $\vec{u}$, $\vec{v}$, & $\vec{w}$

? What if all 3 vectors lie in the same plane?



We can use this fact in reverse to determine whether or not 3 vectors lie in the same plane!

<u>Thm</u>: If $\vec{u}=(u_1,u_2,u_3)$, $\vec{v}=(v_1,v_2,v_3)$, & $\vec{w}=(w_1,w_2,w_3)$ have the same initial point, they lie in the same plane if and only if:

ex// Verify that $(1,2,0)$, $(3,-5,0)$, & $(-7,4,0)$ lie in the same plane.

We first defined vectors simply as directed line segments, & introduced components & coordinate systems later to simplify computations. But a vector has "mathematical existence" even without a coord. system, & a vector's components depend on which coord. system we use!

So since the cross-product was defined in terms of components, could two fixed vectors $\vec{u}$ & $\vec{v}$ have different cross-products in different coord. systems??

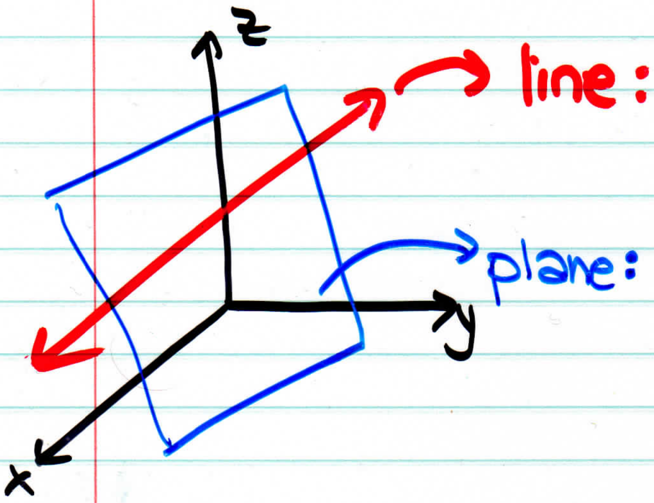
Because $\vec{u}\times\vec{v}$ is completely determined by its direction,

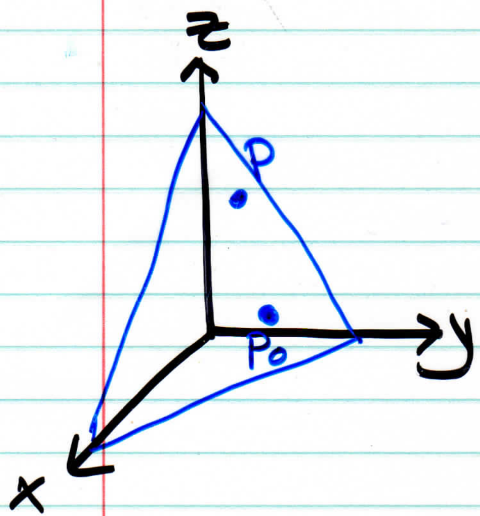
& its length

which do not depend on any coord. system!

ex// Consider 2 perp. vectors of unit length

Section 3.5: Lines & Planes





ex/ Find an eqn' of the plane passing through the point $(1, -2, 3)$ & $\perp$ $\vec{n} = (7, -4, 1)$.

Thm': The graph of the eqn' $ax + by + cz + d = 0$ (where a, b, c, d not all 0) is the plane with normal vector $\vec{n} = (a, b, c)$.

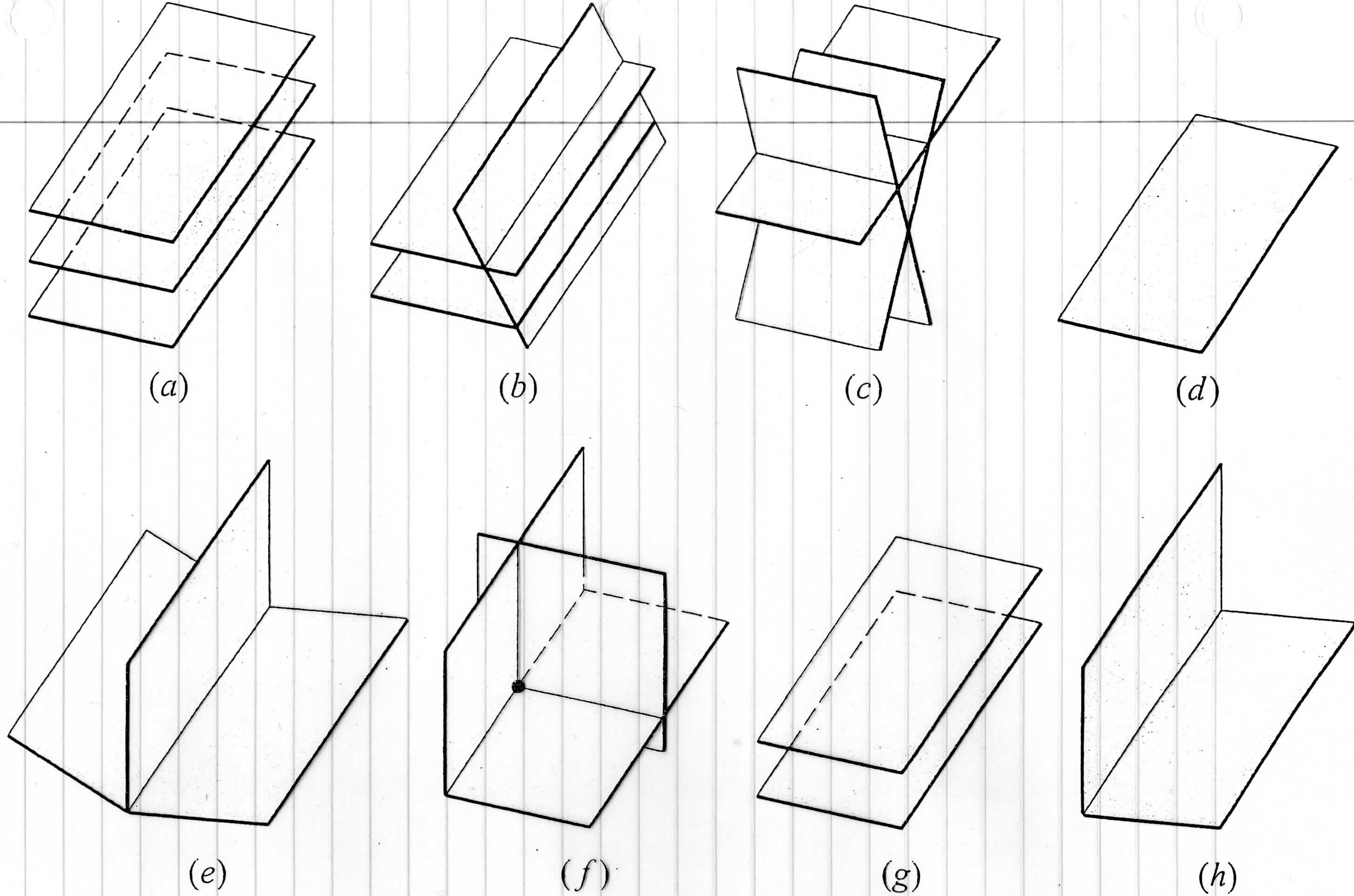


Figure 3.5.2 (*a*) No solutions (3 parallel planes). (*b*) No solutions (2 parallel planes). (*c*) No solutions (3 planes with no common intersection). (*d*) Infinitely many solutions (3 coincident planes). (*e*) Infinitely many solutions (3 planes intersecting in a line). (*f*) One solution (3 planes intersecting at a point). (*g*) No solutions (2 coincident planes parallel to a third plane). (*h*) Infinitely many solutions (2 coincident planes intersecting a third plane).

Just as the solution of a system of linear eqn's

$$\begin{aligned} ax+by &= k_1 \\ cx+dy &= k_2 \end{aligned}$$

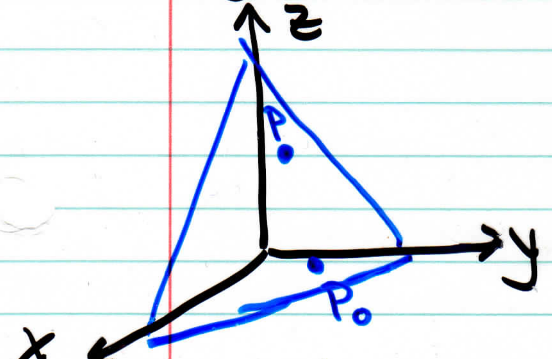
has soln's that correspond to the intersection of the lines...

$$\begin{aligned} ax+by+cz &= k_1 \\ dx+ey+fz &= k_2 \\ gx+hy+iz &= k_3 \end{aligned}$$

corresponds to the intersection of...

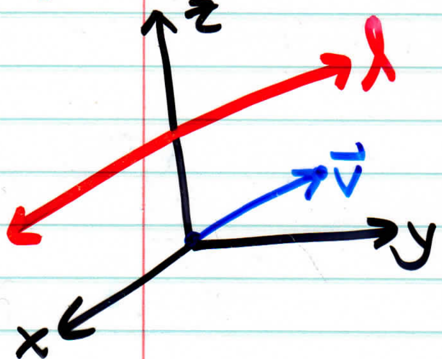
ex// Find the eqn of the plane through the points $P_1(1,2,-1)$, $P_2(2,3,1)$, & $P_3(3,-1,2)$

We can write the P.N. form of a plane nicely in terms of vectors!



ex// Find the vector eqn' of the plane that passes through the point $(6,3,-4)$ & is $\perp$ $\vec{n}=(-1,2,5)$.

? How do we get the eqn' for a line in 3-space?



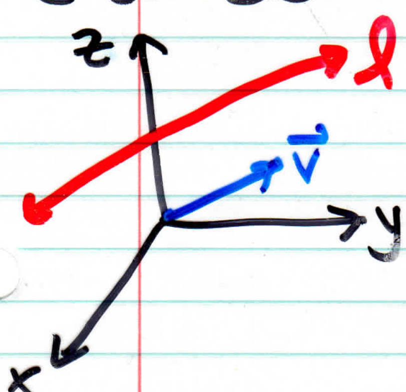
ex// a) Find parametric eqn's for the line l passing through $P_1\ (2,4,-1)$ & $P_2\ (5,0,7)$.

b) where does the line intersect the x-y plane?

ex// Find parametric eqn's for the line of intersection of the planes

$$3x+2y-4z-6=0 \text{ \& } x-3y-2z-4=0.$$

We can also write a line in terms of vector eqns!

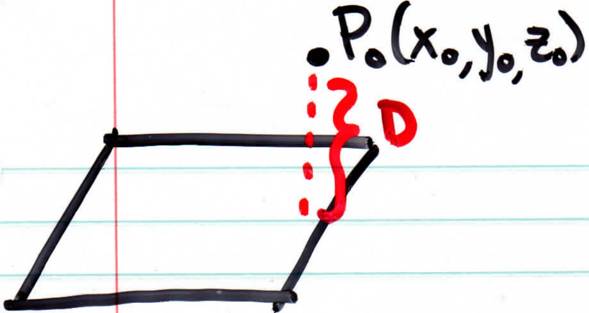


ex// What is the vector form eqn' for the line through $(1,-2,7)$ & parallel to the vector $\vec{v}=(4,0,-3)$.

There are 2 major "distance problems" in 3-space:

a) Find the distance between a point & a plane.

b) Find the distance between two parallel planes.



$\Rightarrow$ The distance b/w a point $P_0(x_0, y_0, z_0)$ & the plane $ax+by+cz+d=0$:

ex// What is the distance between the point $(4, 2, -1)$ & the plane $3x-y+5z=-4$?

ex// a) Verify that the planes $x+2y-2z=3$ & $2x+4y-4z=7$ are parallel.

b) What is the distance between the planes?