Chapter 1: Systems of Linear Equations & Matrices

Section 1.1: Intro. to Systems of Linear Equations

Defn': A "linear equation" in n variables $(x_1, x_2, x_3, \ldots, x_n)$ is one that can be expressed in the form:

ex// Are the following linear eqn's?

a) $x_1 + 2x_2 = 0$

b) $3x + y = 7$

¿ What is a **SOLUTION** to these eqn's ?

It is a sequence of n numbers $(s_1, s_2, \ldots, s_n)$ that satisfies the eqn' when we plug in $x_1 = s_1$, $x_2 = s_2$, ... $x_n = s_n$

ex// Find the solution set of:

a) $x + 3y = 6$

<u>Defn'</u>: A finite set of linear eqn's in the variables $x_1, x_2, \ldots, x_n$ is called a "system of linear equations" & its soln' set $(s_1, s_2, \ldots, s_n)$ would satisfy ALL of the eqn's

A system of linear eqn's can have

1. One (unique) soln'
2. An infinite number of soln's
3. No soln's

Let's illustrate the above by examining 3 systems, each having 2 linear eqn's in unknowns x & y:

ex 3// $x+y=1$
$-7x-7y=-7$

An arbitrary system of m linear eqn's in n unknowns can be written as:

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}
$$

These systems would be difficult/nearly impossible to solve by hand. A computer could do it, but how would we enter the info into the computer??

Augmented Matrix:

ex// write the system below in the form of an augmented matrix:

This can now help us to solve a system of linear eqn's. ? How ?

We replace a given system with a new system that has the same soln! but that is easier to solve!

Elementary Row Operation:

① Multiply an eqn (Row) by a non-zero constant

② Interchange eqn's (Rows)

③ Add a multiple of one eqn (Row) to another.

ex// Solve the system below by using an augmented matrix & row operations.

$$\begin{aligned} 2x - 3y + 3z &= 6 \\ x + 2y - z &= 3 \\ x - y + z &= 2 \end{aligned}$$

Section 1.2: Gaussian Elimination

We can reduce a matrix into "ROW-ECHELON" or "REDUCED ROW-ECHELON" form:

1. If a row does not consist entirely of zeros, then the first non-zero number in the row is a 1 → a "leading 1".
2. If there are any rows that consist entirely of zeros, then they are grouped together at the bottom of the matrix.
3. In any two successive rows that do not consist entirely of zeros, the leading 1 in the lower row occurs farther to the right than the leading 1 in the higher row.
4. Each column that contains a leading 1 has zeros everywhere else in that column.

ex// Are the following in REF, RREF, or neither?

$$\begin{bmatrix} 1 & 3 & -1 & 0 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

If, by a sequence of elementary row operations the augmented matrix for a system is put into RREF (or REF) then the soln' set of the system will become fairly simple to find.

ex// The following are augmented matrices representing linear systems that have been reduced using row op.'s to RREF, solve:

a) $\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 4 \end{bmatrix}$

b) $\begin{bmatrix} 1 & 0 & 0 & 4 & -1 \\ 0 & 1 & 0 & 2 & 6 \\ 0 & 0 & 1 & 3 & 2 \end{bmatrix}$

c) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

ex// Solve the system below using Gaussian Elimination & back substitution:

$$2x + 4y - 3z = 1$$
$$x + y + 2z = 9$$
$$3x + 6y - 5z = 0$$

ex// Solve the systems below using Gauss-Jordan elimination:

a)
$$\begin{aligned} -2x_2 + 3x_3 &= 1 \\ 3x_1 + 6x_2 - 3x_3 &= -2 \\ 6x_1 + 6x_2 + 3x_3 &= 5 \end{aligned}$$

b)
$$\begin{aligned} 4x - 8y &= 12 \\ 3x - 6y &= 9 \\ -2x + 4y &= -6 \end{aligned}$$

A system of linear eqn's is said to be **"homogeneous"** if the constant terms are all zero:

$$\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{12}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0 \\ \quad\vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = 0 \end{array}$$

Every homogeneous system is consistent, since $x_1 = 0,\ x_2 = 0,\ \ldots,\ x_n = 0$ is always a soln'!

So, these systems can have either just the trivial soln', or ∞ soln's (one of which is the trivial one).

ex// Explain the two options above in terms of straight lines (systems of 2 linear eqn's in 2 unknowns).

<u>Theorem</u>: A homogeneous system of linear eqn's with more unknowns than eqn's has ∞ soln's.

ex// Without using pen & paper, determine which of the following systems have non-trivial soln's. Find those soln's using Gaussian Elim.

a) $$\begin{aligned} x_1 + 3x_2 - x_3 &= 0 \\ x_2 - 8x_3 &= 0 \\ 4x_3 &= 0 \end{aligned}$$

b) $$\begin{aligned} 3x_1 - 2x_2 &= 0 \\ 6x_1 - 4x_2 &= 0 \end{aligned}$$

c) $3x_1 + x_2 + x_3 + x_4 = 0$
$5x_1 - x_2 + x_3 - x_4 = 0$

Section 1.3: Matrices & Matrix Operations

Defn: A "matrix" is a rectangular array of numbers, called "entries," & matrices may arise in contexts other than representing systems of linear eqn's.

Its **size** is determined by the numbers of rows & columns it has:

ex// What is the size of the following matrices?

$$\begin{bmatrix} 1 & 2 \\ 3 & 0 \\ -1 & 4 \end{bmatrix} \qquad \begin{bmatrix} 2 & 1 & 0 & -3 \end{bmatrix}$$

$$\begin{bmatrix} e & \pi & -\sqrt{2} \\ 0 & 1/2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 3 \end{bmatrix} \qquad \begin{bmatrix} 4 \end{bmatrix}$$

<u>note</u>: we usually use capital letters as names for matrices:

$$A_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

ex// What is the b_{23} entry of $B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

For a square matrix, we can look at the "main diagonal", whose entries have $i=j$.

$$A_{n\times n} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & & & & \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}$$

We can do some of the same things that we do with numbers, with matrices, but in slightly different ways (add, subtract, mult, etc.).

Defn': Two matrices are said to be "equal" if they have the same sizes & their corresponding entries are equal (ie, $A=B$ only if $a_{ij}=b_{ij}$ for all i & j).

SCALAR MULTIPLICATION: If we multiply a matrix by some constant ("scalar"), ie, cA.

ex// $A=\begin{bmatrix} 2 & -3 & 4 \\ -1 & 3 & 1 \end{bmatrix}$ $B=\begin{bmatrix} 0 & 2 \\ -1 & 3 \end{bmatrix}$, find

a) $2A$

b) $\frac{1}{3}B$

MATRIX ADDITION/SUBTRACTION: Two matrices that are the same size can be added or subtracted by adding/subtracting corresponding entries.

ex// $A = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 3 & 1 \\ 4 & 5 \end{bmatrix}$ $C = \begin{bmatrix} 4 & 0 & 6 \\ 2 & -4 & 8 \end{bmatrix}$

a) Find $A+B$

b) Find $2A-B$

c) Find $\frac{1}{2}C+B$

MATRIX MULTIPLICATION: If A is an $m\times r$ matrix & B is an $r\times n$ matrix, then the *product* AB is the $m\times n$ matrix whose entries are:
To find the entry in row i, column j of AB, single out row i from A & column B from j, multiply the corresponding entries from row & column, & then add up the resulting products.

ex// If $A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix}$ & $B = \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix}$

a) Find the $(2,3)$ entry of AB

b) Find AB

ex// Given the following info., state which operations are well defined, & if so, the size of the result. $A_{3\times3}$ $B_{3\times2}$ $C_{2\times3}$ $D_{2\times3}$

a) $\frac{1}{2}C + D$

b) CD

c) DB

d) BD

e) $3ABC + A$

note: with numbers, order of multiplication DOES NOT MATTER, but it does with matrices!

A matrix can be partitioned into smaller matrices by inserting horizontal/vertical lines between selected rows/columns. This can help us when looking for particular entries in a product matrix.

ex/ $A=\begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix}$ $B=\begin{bmatrix} 4 & 2 \\ 0 & 3 \\ -1 & 5 \end{bmatrix}$

a) Find the 1^{st} row & 2^{nd} column of AB:

b) Find the $(2,1)$ entry of AB.

Row & column matrices provide an alternate way of thinking about matrix multiplication:

$$A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \qquad \vec{x}=\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Matrix multiplication has an important application to systems of linear eqn's:

Consider any system of m linear eqn's in n unknowns:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

We already know 1 way to use matrices to represent this system (augmented matrix), or...

TRANSPOSE: If A is any $m \times n$ matrix, then the "**transpose**" of A, A^T, is the $n \times m$ matrix that results from interchanging the rows & columns of A.

ex// $A = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 3 & 4 \end{bmatrix}$ $\quad B = \begin{bmatrix} 1 & -1 & 5 \\ 6 & 0 & 8 \end{bmatrix}$

Find A^TB

TRACE: If A is a square matrix, then the "**trace**" of A, $tr(A)$, is the sum of the entries on the main diagonal of A.

ex// $C = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $tr(C) = ?$

ex// Find $AB^T - 2C$

Section 1.4: Inverses; Rules of Matrix Arithmetic

We saw in a previous example that though $ab = ba$ for numbers, AB & BA need not be equal (for matrices).

? What other properties do matrices have?

Properties of Matrix Arithmetic

Assuming the sizes of these matrices allow for the operations to be performed...

i) $A+B=B+A$ (commutative law for addition)

ii) $A+(B+C)=(A+B)+C$ (associative law - addition)

iii) $A(BC)=(AB)C$ (associative law - multiplication)

iv) $A(B+C)=AB+AC$ (left distributive law)

v) $(B+C)A=BA+CA$ (right distributive law)

vi) $A(B-C)=AB-AC$

vii) $(B-C)A=BA-CA$

viii) $a(B\pm C)=aB\pm aC$

ix) $(a\pm b)C=aC\pm bC$

x) $a(bC)=(ab)C$

xi) $a(BC)=(aB)C=B(aC)$

Defn!: a matrix full of zeros is a "zero matrix"

Here is another example of a property that holds with numbers, but not with matrices...

- If $ab=ac$ & $a\neq 0$, then $b=c$ (cancellation law)
- If $ad=0$, then $a=0$ or $d=0$ (or both)

Consider

$$A=\begin{bmatrix}0&1\\0&2\end{bmatrix}\quad B=\begin{bmatrix}1&1\\3&4\end{bmatrix}\quad C=\begin{bmatrix}2&5\\3&4\end{bmatrix}\quad D=\begin{bmatrix}3&7\\0&0\end{bmatrix}$$

Some "0" properties do carry over...

Properties of Zero Matrices

i) $A+0=0+A=A$ ii) $0-A=-A$

iii) $A-A=0$ iv) $A\cdot 0=0$ & $0\cdot A=0$

Defn: An $n\times n$ "identity matrix" is a square matrix with 1's on the main diagonal & 0's every where else.

ex// Prove that for any $A_{3\times 3}$, $B_{1\times 3}$, the above holds

Theorem: If R is the RREF of an $n \times n$ matrix A, then either R has a row of zeros, or R is I_n.

Defn: If A is a square matrix & if a matrix B of the same size can be found such that $AB = BA = I$, then A is said to be "invertible" & B is the "inverse" of A.

If no such matrix B can be found, A is "singular."

ex// If $A = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}$ & $B = \begin{bmatrix} 3 & 5 \\ -1 & 3 \end{bmatrix}$, verify that B is the inverse of A

ex// Verify that $A=\begin{bmatrix} -1 & 0 & 3 \\ 1 & 0 & 2 \\ 3 & 0 & 4 \end{bmatrix}$ is singular

Thm': If B & C are both inverses of A, then B=C (ie, the inverse is unique!)

So we give the unique inverse a special name: A^{-1}, & $AA^{-1}=I$, $A^{-1}A=I$

? Can we tell if a matrix will have an inverse ?

Defn: If A is a square matrix, then:

$$A^0 = I, \quad A^n = \underbrace{A \cdot A \cdots A}_{n \text{ times}} \quad (n > 0 \text{ integer})$$

& if A is invertible: $A^{-n} = (A^{-1})^n = \underbrace{A^{-1}A^{-1}A^{-1} \cdots A^{-1}}_{n \text{ times}}$

Laws of Exponents

i) For square matrices, $A^r A^s = A^{r+s}$ & $(A^r)^s = A^{rs}$

ii) For invertible matrices

a) A^{-1} is invertible & $(A^{-1})^{-1} = A$

b) A^n is invertible & $(A^n)^{-1} = (A^{-1})^n$, $n = 0, 1, 2 \ldots$

c) kA is invertible & $(kA)^{-1} = \frac{1}{k} A^{-1}$

ex// Using A & A^{-1} from the last example, find A^3 & A^{-2}

Thm: The matrix $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible if $ad-bc\neq 0$, & the inverse will be:

$$A^{-1}=\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}=\begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}$$

Thm: If A & B are invertible & of the same size, then AB is invertible & $(AB)^{-1}=B^{-1}A^{-1}$
(& this holds for the product of any number of invertible matrices)

ex// Find the inverse of $A=\begin{bmatrix} 2 & 3 \\ 2 & 4 \end{bmatrix}$

We can define a "polynomial fcn" using matrices:

If A is $m \times m$ & $p(x) = a_0 + a_1 x + \cdots + a_n x^n$;

then $p(A) = a_0 I + a_1 A + a_2 A^2 + \cdots + a_n A^n$

ex// If $p(x) = x^2 - 2x + 3$ & $A = \begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix}$, find $p(A)$

Properties of the Transpose

Providing matrix sizes allow for well defined operations

i) $(A^T)^T = A$

ii) $(A \pm B)^T = A^T \pm B^T$

iii) $(kA)^T = kA^T$ (k scalar)

iv) $(AB)^T = B^T A^T$

<u>Thm'</u>: If A is invertible, then so is A^T, &

$$(A^T)^{-1} = (A^{-1})^T$$

ex// For $A = \begin{bmatrix} -3 & 2 \\ 3 & -3 \end{bmatrix}$, find A^{-1}, A^T, $(A^{-1})^T$, & verify that $(A^T)^{-1} = (A^{-1})^T$

ex// Show that $A = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$ is invertible & find its inverse

Section 1.5: Elementary Matrices & a Method for Finding A^{-1}

Defn': An $n \times n$ matrix is called an "elementary matrix" if it can be obtained from the $n \times n$ identity matrix I_n by performing a single row op.

ex// Find a 2×2, 3×3, & 4×4 elementary matrix

Thm': If the elementary matrix E results from performing a certain row op. on I_m & A is $m \times n$, then EA is the matrix that results when the same row op. is performed on E.

If we apply one row op. to an identity matrix to get E, we can apply its "inverse" to get back to I!

Thm': Every Elementary matrix is invertible, & the inverse is also an elementary matrix.

Thm': Equivalent Statements ...

For $A_{n\times n}$, the following are equivalent

i) A is invertible

ii) $A\vec{x}=0$ has only the trivial solution

iii) The RREF of A is I_n

iv) A is expressible as a product of elementary matrices

? How does this help us find a matrix's inverse?

ex// Find the inverse of $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$

Section 1.6: Further Results on Systems of Equations & Invertibilily

Thm: If A is an invertible $n \times n$ matrix, then for each $n \times 1$ matrix b, the system $A\vec{x} = b$ has exactly one solution: $\vec{x} = A^{-1}b$

ex// Using inverses, find the soln' of

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 5 \\ 2x_1 + 5x_2 + 3x_3 &= 3 \\ x_1 \quad\;\; + 8x_3 &= 17 \end{aligned}$$

Sometimes we need to find soln's to a sequence of systems, each of which has the same square coefficient matrix A

We can solve this system using inverses...

ex// Solve the systems

a)
$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 4 \\ 2x_1 + 5x_3 + 3x_3 &= 5 \\ x_1 + 8x_3 &= 9 \end{aligned}$$

b)
$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 1 \\ 2x_2 + 5x_3 + 3x_3 &= 6 \\ x_1 + 8x_3 &= -6 \end{aligned}$$

Thm': Let A be a square matrix, then:

i) If B is square & satisfies $BA = I \Rightarrow B = A^{-1}$

ii) " " " " " " $AB = I \Rightarrow B = A^{-1}$

... Equivalent Statements

v) $A\vec{x} = b$ is consistent for every $n \times 1$ matrix b

vi) $A\vec{x} = b$ has exactly one soln' for every $n \times 1$ matrix b.

So a square matrix that is invertible represents the coefficients of a consistent system, but other systems may or may not be consistent.

ex/ What conditions must b_1, b_2, & b_3 satisfy in order for the following systems to be consistent?

a) $x_1 + x_2 + 2x_3 = b_1$
$x_1 + x_3 = b_2$
$2x_1 + x_2 + 3x_3 = b_3$

$$\text{b)}\quad \begin{aligned} x + y + z &= b_1 \\ x + y - 4z &= b_2 \\ -4x + y + z &= b_3 \end{aligned}$$

Section 1.7: Diagonal, Triangular, and Symmetric Matrices

Defn': A square matrix in which all entries off the main diagonal are zeros is a "diagonal matrix".

$$D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix}_{n \times n}$$

Another shortcut with diagonal matrices:

$$\begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} =$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \\ a_{41} & a_{42} \end{bmatrix}\begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} =$$

ex// for $A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}$ & $B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

find A^{-1}, A^2, A^{-2}, B^{-1}, B^3, B^{-3}

<u>Defn'</u>: A square matrix in which all entries...

... above the main diagonal are zeros is "lower triangular."

... below the main diagonal are zeros is "upper triangular".

Both are called "triangular.

ex// Are the following triangular?

a) $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$ b) $\begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$

d) a square matrix in REF?

<u>Thm'</u>: i) The transpose of a L.T. matrix is L.T.
" " " " U.T. " " U.T.

ii) The product of L.T. matrices is L.T.
" " " U.T. " " U.T.

iii) A triangular matrix is invertible if and only if its diagonal entries are all non-zero

iv) The inverse of an invertible L.T. matrix is L.T.
" " " " " U.T. " " U.T.

ex// $A=\begin{bmatrix} 1 & 3 & -1 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix}$ $B=\begin{bmatrix} 3 & -2 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}$ $C=\begin{bmatrix} 1 & 0 & 0 \\ 4 & -1 & 0 \\ 0 & 3 & 3 \end{bmatrix}$

Find A^T, B^{-1}, C^{-1}, & AB (if defined)

Defn: A square matrix A is "symmetric" if $A=A^T$.
In other words, it is symmetric if
$a_{ij} = a_{ji}$

Thm': If A & B are symmetric matrices with the same size, then

i) A^T is symmetric

ii) A+B & A-B are symmetric

iii) kA is symmetric (for any scalar k)

ex// $A=\begin{bmatrix} 1 & -1 & 4 \\ -1 & 2 & 0 \\ 4 & 0 & 3 \end{bmatrix}$ $B=\begin{bmatrix} 1 & 5 & -1 \\ 5 & 0 & 3 \\ -1 & 3 & 2 \end{bmatrix}$

a) verify that A & B are symmetric

b) verify A+B is symmetric

c) is AB symmetric?

iv) the product of A & A^T is symmetric ex// for $A = \begin{bmatrix} 1 & -2 & 0 \\ -1 & 3 & 2 \end{bmatrix}$, show AA^T & A^TA are symmetric

<u>Thm</u>': If A is an invertible symmetric matrix, then A^{-1} is symmetric.

<u>Thm</u>': If A is an invertible matrix, then AA^T & A^TA are also invertible.