CHAPTER 2: Determinants

Just like we can define a real-valued fcn' $f(x)$ that takes a number, x, & outputs a number, we can similarly define a matrix fcn' $f(X)$ that takes a matrix X & outputs a number – this fcn' is called the "determinant."

Section 2.1: Determinants by Cofactor Expansion

Recall that for a 2x2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, if

$ad-bc \neq 0$, $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$ad-bc$ is the "determinant" of A, $\det(A)$ or $|A|$.

Defn': If A is a square matrix, the "minor" of entry a_{ij} (M_{ij}) is the det. of the submatrix that remains after the i^{th} row & j^{th} column are deleted from A. The number $(-1)^{i+j}M_{ij}$ is the "cofactor" of entry a_{ij} (C_{ij}).

ex// $A = \begin{bmatrix} -3 & 7 & -9 \\ 0 & -1 & 6 \\ 1 & 4 & 2 \end{bmatrix}$, find $M_{11}, C_{11}, M_{32}, C_{32}$

Defn': The det. of a 3x3 matrix is

$\det(A) = a_{11}M_{11} + a_{12}(-M_{12}) + a_{13}M_{13} = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$

This method of evaluating det(A) is "cofactor expansion"

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} =$$

ex/ Find det(A) for $A = \begin{bmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{bmatrix}$

<u>Defn'</u>: The det. of an $n \times n$ matrix A can be computed by multiplying the entries in any row (or column) by their cofactors and adding the resulting products; for each $1 \le i \le n$ & $1 \le j \le n$:

$$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$ (expanding along column j)

& $$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$ (exp. along row i)

We can be smart about our choice of row / column to expand along...

ex// $A = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 3 & 1 & 2 & 2 \\ 1 & 0 & -2 & 1 \\ 2 & 0 & 0 & 1 \end{bmatrix}$

Defn: If A is any $n \times n$ matrix & C_{ij} is the cofactor of a_{ij}, then the "matrix of cofactors from A" is

$$\begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix} \rightarrow$$ its transpose is the "adjoint of A", $\operatorname{adj}(A)$.

ex// Find $\operatorname{adj}(A)$ for $A = \begin{bmatrix} 1 & 0 & 3 \\ -2 & 1 & 4 \\ -3 & 2 & 5 \end{bmatrix}$

Thm: If A is an invertible matrix, then

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)$$

If $i=j$, then ⊛ is the cofactor expansion of $\det(A)$ along the i^{th} row of A. If $i \neq j$ then the a's & cofactors come from different rows of A. If we form a sum ⊛ with a's & cofactors coming from different rows, its value will be 0, (see example 5 of this section). Therefore:

ex// Use this Thm' to find the inverse of

$$A = \begin{bmatrix} 1 & 0 & 3 \\ -2 & 1 & 4 \\ -3 & 2 & 5 \end{bmatrix}$$

Thm': If A is an $n \times n$ triangular matrix (U.T., L.T., or diagonal), then $\det(A)$ is the product of the entries on the main diagonal, $\det(A) = a_{11} a_{22} \cdots a_{nn}$.

Now we can prove that a triangular matrix is invertible if & only if all diagonal entries are non-zero.

There is a third way to find solutions to systems of n eqn's in n unknowns...

Thm' (Cramer's Rule): If $A\vec{x} = b$ is a system of n linear eqn's in n unknowns such that $\det(A) \neq 0$, then the system has a unique soln:

$$x_1 = \frac{\det(A_1)}{\det(A)}, \quad x_2 = \frac{\det(A_2)}{\det(A)}, \quad \ldots, \quad x_n = \frac{\det(A_n)}{\det(A)}$$

where A_j is the matrix $\begin{bmatrix} a_{11} \cdots a_{1j} = b_1 \cdots a_{1n} \\ a_{21} \cdots a_{2j} = b_2 \cdots a_{2n} \\ \vdots \qquad \vdots \\ a_{n1} \cdots a_{nj} = b_n \cdots a_{nn} \end{bmatrix}$

Section 2.2: Evaluating Determinants by Row-Reduction

The determinant of a square matrix can be evaluated by reducing the matrix to row-echelon form.

Thm: Let A be a square matrix. If A has a row or column of zeros, then $\det(A)=0$.

Thm: Let A be a square matrix, then $\det(A)=\det(A^T)$

Thm: Let A be an $n\times n$ matrix:

a) If B is the matrix that results when a single row or column of A is multiplied by a scalar k, then $\det(B)=k\det(A)$.

b) If B is the matrix that results when two rows or columns of A are interchanged, then $\det(B)=-\det(A)$

c) If B is the matrix that results when one row (or column) of A is added to another row (or column) of A, then $\det(B)=\det(A)$.

Thm: If E is an $n \times n$ elementary matrix,

a) If E results from multiplying a row of I_n by k, $\det(E) = k$.

b) " " " " interchanging 2 rows ", $\det(E) = -1$.

c) " " " " adding a multiple of one row of I_n to another, then $\det(E) = 1$.

ex// Use this Thm to evaluate the following:

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} \qquad \begin{vmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}, \qquad \begin{vmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

Thm: If A is a square matrix with two proportional rows (or columns), then $\det(A) = 0$.

ex// Evaluate the determinant of

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 5 & 7 \\ -2 & 4 & 6 \end{bmatrix}$$

? How does all of this help us calculate determinants ?

→ use row op's to reduce to U.T. or L.T.

→ compute the determinant of the U.T. (L.T.) matrix.

→ using row (or column) op's, relate the det of the triangular matrix to that of the original!

ex// evaluate det(A) where $A = \begin{bmatrix} 0 & 1 & 3 \\ 2 & -2 & 4 \\ -3 & 5 & 1 \end{bmatrix}$

ex// Compute the determinant of $A = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 2 & 7 & 0 & 6 \\ 0 & 6 & 3 & 0 \\ -1 & 3 & 1 & -5 \end{bmatrix}$

Or, we can do a combination of cofactor expansion & row op's!

ex// evaluate $\begin{vmatrix} 3 & 5 & -2 & 6 \\ 1 & 2 & -1 & 1 \\ 2 & 4 & 1 & 5 \\ 3 & 7 & 5 & 3 \end{vmatrix}$

Section 2.3: Properties of the Determinant

We saw in the last section that $\det(kA) = k^n \det(A)$ for an $n \times n$ matrix A. In this section, we will explore more such properties.

ex// Does $\det(A+B) = \det(A) + \det(B)$?

Thm: Let A, B, and C be $n \times n$ matrices that differ only in a single row, (the i^{th}), and assume that the i^{th} row of C can be obtained by adding corresponding entries in the i^{th} rows of A & B, then $\det(C) = \det(A) + \det(B)$ (& same holds for columns).

ex// Use the Thm' above to calculate $\det(A+B)$ for $A = \begin{bmatrix} 1 & 9 & 9 \\ -1 & 0 & 3 \\ 2 & 4 & 5 \end{bmatrix}$ & $B = \begin{bmatrix} 7 & -9 & -9 \\ -1 & 0 & 3 \\ 2 & 4 & 5 \end{bmatrix}$

? Does $\det(AB) = \det(A)\det(B)$?

<u>Thm'1</u>: If B is an $n \times n$ matrix, and E is an $n \times n$ elementary matrix, then

$$\det(EB) = \det(E)\det(B)$$

<u>Proof)</u> If E results from multiplying a row of I_n by k, then EB results from B by multiplying a row by k, so

<u>Thm'2</u>: A square matrix A is invertible if and only if $\det(A) \neq 0$.

<u>Proof)</u> Let R be the RREF of A, then either both $\det(A)$ & $\det(R)$ are 0, or both are non-0 because

If A is invertible, our "Equivalent Statements" Thm' says $R=I$, so

Note: It follows from this Thm' that a square matrix with 2 proportional rows (or columns) is not invertible:

Thm': If A & B are square matrices of the same size, then $\det(AB)=\det(A)\det(B)$

Proof) If A is not invertible, then neither is AB, & thus ex// Verify the Thm' for $A=\begin{bmatrix} 1 & 0 & -1 \\ 2 & -4 & 6 \\ 0 & -5 & 3 \end{bmatrix}$ & $B=\begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 7 \\ -1 & 0 & 4 \end{bmatrix}$

<u>Thm'</u>: If A is invertible, then $\det(A^{-1})=\dfrac{1}{\det(A)}$

ex// If $A=\begin{bmatrix} 2 & 0 & 0 \\ 19 & 3 & 0 \\ 52 & 76 & -4 \end{bmatrix}$, find $\det(A^{-1})$

ex// For $A_{4\times4}$ & $B_{4\times4}$, $\det(A)=9$ & $\det(B)=-7$, find: