Section 5.5: Row Space, Column Space, & Nullspace

Defn': For an $m \times n$ matrix, A, we can form "row vectors" & "column vectors".

$$A_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

ex// Write the row & column vectors of

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 3 & -1 & 4 \end{bmatrix}$$

Defn': If A is an $m \times n$ matrix, then the subspace of $\mathbb{R}^n$ spanned by the row vectors of A is the "row space" of A, & the subspace of $\mathbb{R}^m$ spanned by the column vectors of A is the "column space" of A. The soln' space of the system $A\vec{x} = \vec{0}$, a subspace of $\mathbb{R}^n$, is the "nullspace" of A.

? Are the rowspace, column space, & nullspace related?

- Are they related to soln's of $A\vec{x}=\vec{b}$?

<u>Thm</u>: A system of linear eqn's $A\vec{x}=\vec{b}$ is consistent if & only if $\vec{b}$ is in the column space of A.

ex// Given $\begin{bmatrix} -1 & 3 & 2 \\ 1 & 2 & -3 \\ 2 & 1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 1 \\ -9 \\ -3 \end{bmatrix}$

a) Show that $\vec{b}$ is in the column space of A.

b) express $\vec{b}$ as a linear combo. of A's column vectors.

Thm: If $\vec{x}_0$ denotes any single soln' of a consistent linear system $A\vec{x}=\vec{b}$, & if $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ form a basis for the nullspace of A (ie, the solution space of $A\vec{x}=\vec{0}$), then every soln' of $A\vec{x}=\vec{b}$ can be expressed in the form:

$$\vec{x} = x_0 + c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k$$

& for all scalars $c_1, c_2, \ldots, c_k$, $\vec{x}$ will be a soln' of $A\vec{x}=\vec{b}$.

ex// For the following system, find a particular soln', & the general soln's of $A\vec{x}=\vec{b}$ & $A\vec{x}=\vec{0}$.

Thm': Elementary row operations do not change the nullspace of a matrix.

ex// Find a basis for the nullspace of

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & 4 & -2 \end{bmatrix}$$

Thm': Elementary row op's do not change the rowspace of a matrix.

ex// Find a basis for the rowspace of

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$$

__BUT__ elementary row op's __will__ change the column space of a matrix!

ex// Look at the previous matrix . . .

__Thm__: If A & B are row equivalent matrices, then

a) A given set of column vectors of A is linearly independent if & only if the corresponding column vectors of B are linearly independent.

b) A given set of column vectors of A forms a basis for the column space of A if & only if the corresponding column vectors of B form a basis for the column space of B.

? Is there a way to find a basis for the row/column space of a matrix ?

<u>Thm'</u>: If a matrix R is in REF, then the row vectors with the leading 1's (ie, the non-zero row vectors) form a basis for the row space of R, & the column vectors with the leading 1's (ie, the non free-variable columns) form a basis for the column space of R.

ex// Find bases for the row/column space of A:

a) $A = \begin{bmatrix} 1 & -2 & 5 & 0 & 3 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

b) $A = \begin{bmatrix} 1 & 4 & -2 \\ 2 & 9 & -1 \\ 2 & 9 & -1 \end{bmatrix}$

ex// Find a basis for the space spanned by

$\vec{v}_1 = (1, -2, 0, 0, 3)$ $\vec{v}_2 = (2, -5, -3, -2, 6)$

$\vec{v}_3 = (0, 5, 15, 10, 0)$ $\vec{v}_4 = (2, 6, 18, 8, 6)$

ex// Find a basis for the row space of A consisting entirely of row vectors from A

$$A = \begin{bmatrix} 1 & -2 & 0 \\ 2 & -5 & -3 \\ 0 & 5 & 15 \end{bmatrix}$$