Section 5.3: Linear Independence

We learned that $S = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_r\}$ spans V if EVERY vector in V is expressible as a linear combo. of the vectors in S. There may be more than one way to do this, but there are conditions under which each vector in V is expressible as a linear combo. of the vectors in S in EXACTLY ONE way.

Defn': If $S = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_r\}$ is a non-empty set of vectors, then $k_1\vec{v}_1 + k_2\vec{v}_2 + \cdots + k_r\vec{v}_r = \vec{0}$ has at least one soln', $k_1 = 0, k_2 = 0, \dots, k_r = 0$.

If this is the only soln' $\Rightarrow$ S is "linearly independent."

If there are other soln's $\Rightarrow$ S is "linearly dependent."

ex// Is the set $\vec{v}_1 = (2, -1, 0, 3)$, $\vec{v}_2 = (1, 2, 5, -1)$, $\vec{v}_3 = (7, -1, 5, 8)$ linearly indep. or dep. ?

For $p_1 = 1-x$, $p_2 = 5+3x-2x^2$, $p_3 = 1+3x-x^2$, $3p_1 - p_2 + 2p_3 = 0 \ni S = \{p_1, p_2, p_3\}$ is also linearly dep.

In fact, the polynomials $1, x, x^2, \dots, x^n$ form a linearly independent set of vectors in P_n.

ex// Do $\vec{i}, \vec{j}, \vec{k}$ form a linearly indep. or dep. set in $\mathbb{R}^3$?

__Note__: vectors $e_1 = (1,0,\dots,0)$, $e_2 = (0,1,0,\dots,0)$, $\dots$, $e_n = (0,\dots,0,1)$ form a linearly indep. set in $\mathbb{R}^n$.

ex// Does $S = \{(4,-2), (12,-6)\}$ form a linearly indep. or dep. set in $\mathbb{R}^2$?

? Why do we say *linearly dependent*?

Thm!: A set S with two or more vectors is;

a) Linearly dep. if & only if at least one of the vectors in S is expressible as a linear combo. of the other vectors in S.

b) Linearly indep. if & only if no vector in S is expressible as a linear combo. of the other vectors in S.

Thm': a) A finite set of vectors that contains the zero vector IS ALWAYS linearly dep.

b) A set with exactly 2 vectors is linearly indep. if & only if neither vector is a scalar multiple of the other.

ex// Are the following linearly dep. or indep.?

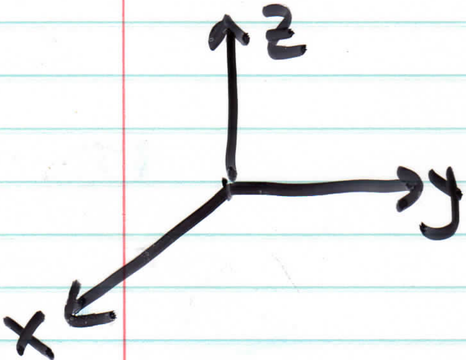
a) $\vec{v}_1 = (1,2,3,4)$, $\vec{v}_2 = (3,-1,7,0)$, $\vec{v}_3 = (0,0,0,0)$

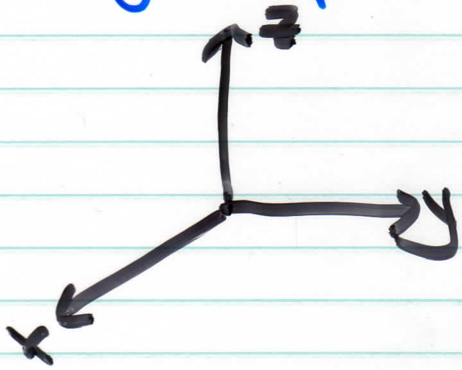
b) $\vec{v}_1 = (1,0,-4,2)$, $\vec{v}_2 = (2,1,-8,4)$

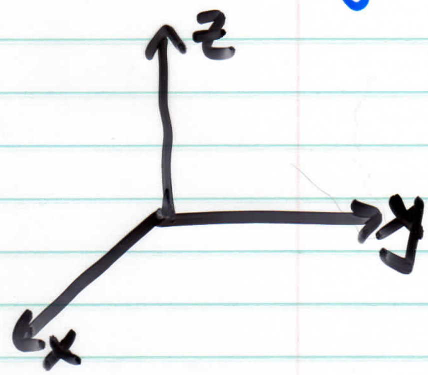
c) $\vec{v}_1 = (1,0,-4,2)$, $\vec{v}_2 = (2,0,-8,4)$

Linear independence has some useful geometric interpretations in $\mathbb{R}^2$ & $\mathbb{R}^3$.

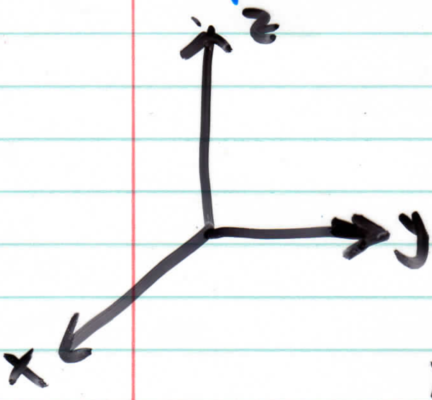
- In $\mathbb{R}^2$ or $\mathbb{R}^3$, a set of 2 vectors is linearly indep. if & only if the vectors do not lie on the same line when they are placed at the origin.

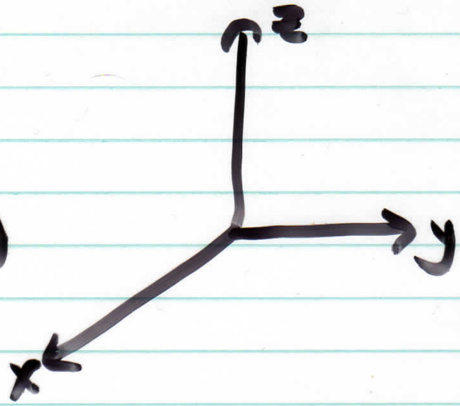


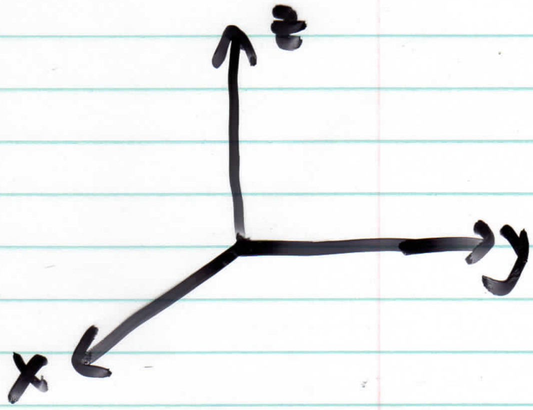




- In $\mathbb{R}^3$, a set of 3 vectors is linearly indep. if & only if the vectors do not all lie in the same plane when they are placed at the origin.







Thm': Let $S = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_r\}$ be a set of vectors in $\mathbb{R}^n$. If $r > n$, then S is linearly dependent.

ex// Is $S = \{(1,2,3), (-2,4,9), (3,1,-7), (4,-1,1)\}$ linearly dep. or indep. in $\mathbb{R}^3$?