Section 5.2: Subspaces

It is possible for one vector space to be contained in another, for example, planes through the origin are a vector space that are contained in the vector space $\mathbb{R}^3$.

Defn': A subset W of a vector space V is called a "subspace" of V if W is itself a vector space under the addition & scalar mult. defined on V.

W "inherits" axioms 2, 3, 4, 5, 7, 8, 9 & 10!

Thm': If W is a set of one or more vectors from a vector space V, then W is a subspace of V if & only if:

a) If $\vec{u}$ & $\vec{v}$ are vectors in W, then $\vec{u}+\vec{v}$ is in W.

b) If k is any scalar & $\vec{u}$ is any vector in W, then $k\vec{u}$ is in W.

ex// Show that a line through the origin of $\mathbb{R}^3$ is a subspace of $\mathbb{R}^3$.

ex// Let W be the set of all points (x,y) in $\mathbb{R}^2$ such that $x \geq 0$, $y \geq 0$. Is W a supspace of $\mathbb{R}^2$?

<u>Note</u>: Every nonzero vector space V has at least two subspaces → V itself, & $\{\vec{0}\}$, the "zero subspace".

ex// List the subspaces of $\mathbb{R}^2$ & $\mathbb{R}^3$ that we have seen so far.

ex// List some subspaces of M_{mn}.

Polynomials of degree $\leq n$ (n a positive integer) is a subspace of $F(-\infty,\infty)$, P_n .

All continuous fcn's also form a subspace of $F(-\infty,\infty)$, & we denote this subspace as $C(-\infty,\infty)$.

If $A\vec{x}=\vec{b}$ is a system of linear eqn's, then each vector $\vec{x}$ that satisfies the eqn' is called a "solution vector" & all of these make up the "solution space".

<u>Thm'</u>: If $A\vec{x}=\vec{0}$ is a homogeneous linear system of m eqn's in n unknowns, then the set of solution vectors is a subspace of $\mathbb{R}^n$.

ex/ Describe the solution space of the following

a) $\begin{bmatrix} 1 & -2 & 3 \\ 2 & -4 & 6 \\ 3 & -6 & 9 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow$ soln' $\begin{aligned} x &= 2s-3t \\ y &= s \\ z &= t \end{aligned}$

b) $\begin{bmatrix} 1 & -2 & 3 \\ -3 & 7 & -8 \\ -2 & 4 & -6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow$ soln': $x=-5t$, $y=-t$, $z=t$

c) $\begin{bmatrix} 1 & -2 & 3 \\ -3 & 7 & -8 \\ 4 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow$ soln': $x=0$, $y=0$, $z=0$

d) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow$ soln': $x=r$, $y=s$, $z=t$

Defn': A vector $\vec{w}$ is called a "linear combination" of the vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_r$ if it can be expressed in the form:

$$\vec{w} = k_1\vec{v}_1 + k_2\vec{v}_2 + \cdots + k_r\vec{v}_r \quad (k\text{'s scalars})$$

note 2: We have already seen that all vectors in $\mathbb{R}^3$ are linear comb.'s of $\vec{i}$, $\vec{j}$, & $\vec{k}$:

ex// Consider the vectors $\vec{u} = (1, 2, -1)$ & $\vec{v} = (6, 4, 2)$ in $\mathbb{R}^3$.

a) Show that $\vec{w} = (9, 2, 7)$ is a linear comb. of $\vec{u}$ & $\vec{v}$.

b) Show that $\vec{w}' = (4, -1, 8)$ is not a linear comb. of $\vec{u}$ & $\vec{v}$.

So for $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_r$ in V, some vectors in V may be linear comb.'s of these, other may not.

Thm: If $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_r$ are vectors in a vector space V, then:

a) The set W of all linear comb.'s of $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_r$ is a subspace of V.

b) W is the smallest subspace of V that contains $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_r$ in the sense that every other subspace of V that contains $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_r$ must contain W.

Defn: If $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r\}$ is a set of vectors in a vector space V, then the subspace W of V consisting of all linear comb's of the vectors in S is called the "space spanned" by $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r$, & we say that these vectors "span" W.

$$W = \text{span}(S) = \text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r\}$$

ex// If $\vec{v}_1$ & $\vec{v}_2$ are noncollinear vectors in $\mathbb{R}^3$ with their initial points at the origin, $\text{span}\{\vec{v}_1, \vec{v}_2\} = ?$

The polynomials $1, x, x^2, \ldots, x^n$ span the vector space P_n since each polynomial

$$p = a_0 + a_1 x + \cdots + a_n x^n$$

$\Rightarrow$

ex// Determine whether the following span $\mathbb{R}^3$

a) $\vec{v}_1 = (2,2,2)$, $\vec{v}_2 = (0,0,3)$, $\vec{v}_3 = (0,1,1)$

b) $\vec{v}_1 = (1,1,2)$, $\vec{v}_2 = (1,0,1)$, $\vec{v}_3 = (2,1,3)$

<u>Note</u>: Spanning sets are not unique: Look at our examples of vectors that span planes & lines - any nonzero vectors on that plane/ line will span the plane/ line!

<u>Thm</u>: If $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r\}$ & $S' = \{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k\}$ are two sets of vectors in vector space V, then
$$\text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r\} = \text{span}\{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k\}$$
if & only if each vector in S is a linear comb. of those in S' & each vector in S' is a linear comb. of those in S.

ex// Use this Thm' to show that $\vec{v}_1 = (0,2)$, $\vec{v}_2 = (3,-1)$ & $\vec{w}_1 = (-3,3)$, $\vec{w}_2 = (6,-4)$ span the same subspace of $\mathbb{R}^2$.