Chapter 5: General Vector Spaces

In this chapter, we generalize the concept of a "vector" even further.

The theory in this chapter will provide a tool for extending our geometric visualization to a wide range of problems where geometric intuition is no longer available.

We will give a set of "axioms" which will class objects as vectors (or not).

Section 5.1: Real Vector Spaces

The following definition consists of ten axioms that set out the "rules of the game"

Defn: Let V be an arbitrary, non-empty set of objects on which two operations are defined:

Addition – associating each pair of objects $\vec{u}$ & $\vec{v}$ in V an object, $\vec{u}+\vec{v}$, the SUM of $\vec{u}$ & $\vec{v}$.

Scalar Multiplication – associating each scalar k & each object $\vec{u}$ in V an object $k\vec{u}$, the SCALAR MULTIPLE of $\vec{u}$ by k.

If the following axioms are satisfied by all objects $\vec{u}$, $\vec{v}$, & $\vec{w}$ in V & all scalars k & m, then we call V a "vector space" & we call the objects in V "vectors".

1. If $\vec{u}$ & $\vec{v}$ are in V, then $\vec{u}+\vec{v}$ is in V
2. $\vec{u}+\vec{v}=\vec{v}+\vec{u}$
3. $\vec{u}+(\vec{v}+\vec{w})=(\vec{u}+\vec{v})+\vec{w}$
4. There is an object $\vec{0}$ in V, the "zero vector" for V, such that $\vec{0}+\vec{u}=\vec{u}+\vec{0}=\vec{u}$ for all $\vec{u}$ in V.
5. For each $\vec{u}$ in V, there is an object $-\vec{u}$ in V, called a "negative" of $\vec{u}$, such that $\vec{u}+(-\vec{u})=(-\vec{u})+\vec{u}=\vec{0}$.
6. If k is any scalar & $\vec{u}$ is any object in V, then $k\vec{u}$ is in V.
7. $k(\vec{u}+\vec{v})=k\vec{u}+k\vec{v}$
8. $(k+m)\vec{u}=k\vec{u}+m\vec{u}$
9. $k(m\vec{u})=(km)\vec{u}$
10. $1\vec{u}=\vec{u}$

$V = \mathbb{R}^n$ is a vector space because it has addition & scalar multiplication (the ones we know), & all of the axioms are satisfied.

ex// Let V = all 2x2 matrices, with additon & scalar mult. as defined in Ch 1 for matrices, is V a vector space?

The set of all real-valued functions with domain $(-\infty,\infty)$ is also a vector space, $F(-\infty,\infty)$.

Every plane in $\mathbb{R}^3$ through the origin is also a vector space (with standard addition & scalar mult.

We know $\mathbb{R}^3$ is itself a vector space, so we just need to define/check Axioms 1, 4, 5, & 6:

ex// If we let $V = \mathbb{R}^2$, but define scalar mult. & addition as follows: $\vec{u} = (u_1, u_2)$, $\vec{v} = (v_1, v_2)$

$\vec{u} + \vec{v} = (u_1 + v_1, u_2 + v_2)$

$k\vec{u} = (ku_1, 0)$, is V a vector space?

If we let V consist of a single object, $\vec{0}$, & define $\vec{0}+\vec{0}=\vec{0}$ & $k\vec{0}=\vec{0}$, then V is a vector space—the "zero vector space".

Thm: Let V be a vector space, $\vec{u}$ a vector (object) in V, & k a scalar, then:

a) $0\vec{u}=0$

b) $k0=0$

c) $-1(\vec{u})=-\vec{u}$

d) if $k\vec{u}=0 \Rightarrow k=0$ or $\vec{u}=0$