Chapter 4: Euclidean Vector Spaces

Section 4.1: Euclidean n-space

We can extend all of our results about vectors in 2- or 3-space to n-space!

(But we no longer have geometric representation).

Equality: $\vec{u}=(u_1,u_2,\ldots,u_n)$ & $\vec{v}=(v_1,v_2,\ldots,v_n)$
if $u_1=v_1,\ u_2=v_2,\ \ldots,\ u_n=v_n$

Sum: $\vec{u}+\vec{v}=(u_1+v_1,u_2+v_2,\ldots,u_n+v_n)$

Scalar Multiple: $k\vec{u}=(ku_1,ku_2,\ldots,ku_n)$

Zero Vector: $\vec{0}=(0,0,\ldots,0)$

Negative: $-\vec{u}=(-u_1,-u_2,\ldots,-u_n)$

Difference: $\vec{v}-\vec{u}=\vec{v}+(-\vec{u})=(v_1-u_1,v_2-u_2,\ldots,v_n-u_n)$

The Properties of Vectors in $\mathbb{R}^n$ are the same as the Properties of Vectors in 2- or 3-space (see section 3.2)

We can also generalize the notions of distance, norm, & angle to $\mathbb{R}^n$.

<u>Euclidean Inner Product</u>: For two vectors $\vec{u}$ & $\vec{v}$ in $\mathbb{R}^n$, $\vec{u}\cdot\vec{v} = u_1v_1 + u_2v_2 + \cdots + u_nv_n$

(Same properties as Dot Product - see 3.3).

ex// For some vectors $\vec{u}$ & $\vec{v}$ in $\mathbb{R}^n$, calculate $(3\vec{u}+2\vec{v})\cdot(4\vec{u}+\vec{v})$

<u>Euclidean Norm/Length</u>: For a vector $\vec{u}$ in $\mathbb{R}^n$,

$$\|\vec{u}\| = (\vec{u}\cdot\vec{u})^{1/2} = \sqrt{u_1^2+u_2^2+\ldots+u_n^2}$$

<u>Euclidean Distance</u>: between points $u=(u_1,u_2,\ldots,u_n)$ & $v=(v_1,v_2,\ldots,v_n)$ in $\mathbb{R}^n$ is:

$$d(u,v) = \|u-v\| = \sqrt{(u_1-v_1)^2+(u_2-v_2)^2+\cdots+(u_n-v_n)^2}$$

ex// Find the distance between $\vec{u}=(1,3,-2,7)$ & $\vec{v}=(0,7,2,2)$

Properties of Length in $\mathbb{R}^n$:

If $\vec{u}$ & $\vec{v}$ are vectors in $\mathbb{R}^n$ & k is a scalar

a) $\|\vec{u}\| > 0$

b) $\|\vec{u}\| = 0$ if & only if $\vec{u} = \vec{0}$

c) $\|k\vec{u}\| = |k|\,\|\vec{u}\|$

d) $\|\vec{u}+\vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$
(Triangle Inequality)

Cauchy-Schwartz Inequality: For $\vec{u}$ & $\vec{v}$ in $\mathbb{R}^n$,

$$|\vec{u}\cdot\vec{v}| \leq \|\vec{u}\|\,\|\vec{v}\|$$

Properties of Distance in $\mathbb{R}^n$:

For $\vec{u}$, $\vec{v}$, & $\vec{w}$ in $\mathbb{R}^n$, & k any scalar,

a) $d(\vec{u},\vec{v}) \geq 0$

b) $d(\vec{u},\vec{v}) = d(\vec{v},\vec{u})$

c) $d(\vec{u},\vec{v}) = 0$ if & only if $\vec{u} = \vec{v}$

d) $d(\vec{u},\vec{v}) \leq d(\vec{u},\vec{w}) + d(\vec{w},\vec{v})$ (Triangle Inequality)

Thm!: For $\vec{u}$ & $\vec{v}$ in $\mathbb{R}^n$, then

$$\vec{u}\cdot\vec{v} = \frac{1}{4}\|\vec{u}+\vec{v}\|^2 - \frac{1}{4}\|\vec{u}-\vec{v}\|^2$$

We still say that $\vec{u}$ & $\vec{v}$ are orthogonal in $\mathbb{R}^n$ if $\vec{u}\cdot\vec{v} = 0$.

Pythagorean Thm' in $\mathbb{R}^n$:

If $\vec{u}$ & $\vec{v}$ are orthogonal, $\|\vec{u}+\vec{v}\| = \|\vec{u}\|^2 + \|\vec{v}\|^2$

Alternative Notation for Vectors in $\mathbb{R}^n$:

We can write vectors in $\mathbb{R}^n$ as row/column matrices...

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \text{ or } \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix}$$

which gives us a nice way to write the dot product!

ex// For $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$ & $\vec{v} = \begin{bmatrix} -1 \\ 0 \\ 4 \\ 3 \\ 2 \end{bmatrix}$, write $\vec{u} \cdot \vec{v}$

If A is $n \times n$, then:

$A\vec{u} \cdot \vec{v} =$

& $\vec{u} \cdot A\vec{v} =$

Matrix Multiplication: $A_{m \times r}$ & $B_{r \times n}$, the i,j^{th} entry of AB:

$$a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \cdots + a_{ir}b_{rj}$$

& $A\vec{x} = \vec{b}$ looks like $\begin{bmatrix} \vec{r}_1 \cdot \vec{x} \\ \vec{r}_2 \cdot \vec{x} \\ \vdots \\ \vec{r}_m \cdot \vec{x} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$

ex// Write the system below in dot product form

$$3x_1 - 4x_2 + x_3 = 1$$
$$2x_1 - 7x_2 - 4x_3 = 5$$
$$x_1 + 5x_2 - 8x_3 = 0$$