Section 2.2: Evaluating Determinants by Row-Reduction

The determinant of a square matrix can be evaluated by reducing the matrix to row-echelon form.

Thm: Let A be a square matrix. If A has a row or column of zeros, then $\det(A)=0$.

Thm: Let A be a square matrix, then $\det(A)=\det(A^T)$

Thm: Let A be an $n \times n$ matrix:

a) If B is the matrix that results when a single row or column of A is multiplied by a scalar k, then $\det(B)=k\det(A)$.

b) If B is the matrix that results when two rows or columns of A are interchanged, then $\det(B)=-\det(A)$

c) If B is the matrix that results when one row (or column) of A is added to another row (or column) of A, then $\det(B)=\det(A)$.

Thm': If E is an $n \times n$ elementary matrix,

a) If E results from multiplying a row of I_n by k, $\det(E) = k$.

b) " " " " interchanging 2 rows ", $\det(E) = -1$.

c) " " " " adding a multiple of one row of I_n to another, then $\det(E) = 1$.

ex// Use this Thm to evaluate the following:

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} \qquad \begin{vmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}, \qquad \begin{vmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

Thm': If A is a square matrix with two proportional rows (or columns), then $\det(A) = 0$.

ex// Evaluate the determinant of

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 5 & 7 \\ -2 & 4 & 6 \end{bmatrix}$$

? How does all of this help us calculate determinants ?

→use row op's to reduce to U.T. or L.T.
→compute the determinant of the U.T. (L.T.) matrix.
→using row (or column) op's, relate the det of the triangular matrix to that of the original!

ex// evaluate det(A) where $A = \begin{bmatrix} 0 & 1 & 3 \\ 2 & -2 & 4 \\ -3 & 5 & 1 \end{bmatrix}$

ex// Compute the determinant of $A = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 2 & 7 & 0 & 6 \\ 0 & 6 & 3 & 0 \\ -1 & 3 & 1 & -5 \end{bmatrix}$

Or, we can do a combination of cofactor expansion & row op's!

ex// evaluate $\begin{vmatrix} 3 & 5 & -2 & 6 \\ 1 & 2 & -1 & 1 \\ 2 & 4 & 1 & 5 \\ 3 & 7 & 5 & 3 \end{vmatrix}$

Section 2.3: Properties of the Determinant

We saw in the last section that $\det(kA) = k^n \det(A)$ for an $n \times n$ matrix A. In this section, we will explore more such properties.

ex// Does $\det(A+B) = \det(A) + \det(B)$?

Thm: Let A, B, and C be $n \times n$ matrices that differ only in a single row, (the i^{th}), and assume that the i^{th} row of C can be obtained by adding corresponding entries in the i^{th} rows of A & B, then $\det(C) = \det(A) + \det(B)$ (& same holds for columns).

ex// Use the Thm above to calculate $\det(A+B)$

for $A = \begin{bmatrix} 1 & 9 & 9 \\ -1 & 0 & 3 \\ 2 & 4 & 5 \end{bmatrix}$ & $B = \begin{bmatrix} 7 & -9 & -9 \\ -1 & 0 & 3 \\ 2 & 4 & 5 \end{bmatrix}$

? Does det(AB) = det(A)det(B) ?

Thm'1: If B is an nxn matrix, and E is an nxn elementary matrix, then

$$\det(EB) = \det(E)\det(B)$$

Proof) If E results from multiplying a row of I_n by k, then EB results from B by multiplying a row by k, so

Thm'2: A square matrix A is invertible if and only if $\det(A) \neq 0$.

Proof) Let R be the RREF of A, then either both det(A) & det(R) are 0, or both are non-0 because

If A is invertible, our "Equivalent Statements" Thm' says $R=I$, so

Note: It follows from this Thm' that a square matrix with 2 proportional rows (or columns) is not invertible:

Thm': If A & B are square matrices of the same size, then $\det(AB) = \det(A)\det(B)$

Proof) If A is not invertible, then neither is AB, & thus ex// Verify the Thm' for $A=\begin{bmatrix} 1 & 0 & -1 \\ 2 & -4 & 6 \\ 0 & -5 & 3 \end{bmatrix}$ & $B=\begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 7 \\ -1 & 0 & 4 \end{bmatrix}$

Thm': If A is invertible, then $\det(A^{-1})=\dfrac{1}{\det(A)}$

ex// If $A=\begin{bmatrix} 2 & 0 & 0 \\ 19 & 3 & 0 \\ 52 & 76 & -4 \end{bmatrix}$, find $\det(A^{-1})$

ex// For $A_{4\times4}$ & $B_{4\times4}$, $\det(A)=9$ & $\det(B)=-7$, find: