Chapter 3: NON-EUCLIDEAN GEOMETRIES

Section 3.1: Background & Some History

Recall the 5 Axioms that we learned in Ch. 1. The 5^{th} stated that:

5 For every line l & every point P not on l, there exists a unique line m through P & parallel to l.

For 2000 years many educated minds thought that the fifth postulate could be deduced from the first four - but could not do it! Then others began to look in the opposite direction: perhaps the 5^{th} is <u>independent</u> from the other four & therefore is not deducible from them.

In the 19^{th} century, a mathematician named Nikolai Lobachevsky discussed a new geometry that is now called "hyperbolic geometry".

In this new geometry, Euclid's fifth axiom is NOT true while the other four still are.

<u>Hyperbolic 5th Axiom</u>: For every line l & every point P that does not lie on l, there exists <u>infinitely many lines</u> through P that are parallel to l.

To understand how this is possible we need to rethink our idea of the world as we see it...

Lets think of this new world as existing on a crystal ball...

Our usual concept of (Euclidean) distance does not describe the geometry of the world on the crystal ball. This example is NOT a model of hyperbolic geometry, but it is a good introduction to it.

Section 3.2: Hyperbolic Geometry

We learned in our crystal ball example that lines could appear curved to us, & that for some spaces the usual notion of distance does not apply.

Poincaré Model of Hyperbolic Geometry:
Consists of all of the points in the interior of a circle. The circle itself plays the role of the horizon. (H)

We need to define *lines* in this space. The first type of "hyperbolic lines" in this model consist of all open (no end points) line segments along the diameters of the circle.

The second type consists of all open circular arcs within the horizon.

These circles should be perpendicular to H: the angle (θ) between two circles intersecting at a point A is the same as the angle b/w the two radial segments starting at the centers of the circles & ending at A.

We could prove that the first four of Euclid's axioms hold in this space by defining things like "hyperbolic length" - but we will focus on the 5th:

Some other properties in the classical, Euclidean geometry also fail in the setting of hyperbolic geometry.

For example, it follows from the 5th axiom that the sum of the interior angles of any triangle is 180°. Since the fifth axiom does NOT hold in hyperbolic geometry - this property also fails.

This interesting property has the consequence that we will have infinitely many <u>regular monohedral</u> tilings of the hyperbolic plane!

Section 3.3: Basic Constructions in the Poincare Model of Hyp. Geometry

ex.1// Construct a hyperbolic line through any given point:

ex 2// Construct the unique hyperbolic line through two given points.

Section 3.4: Tiling of the Hyperbolic Plane

Recall: the sum of the angles in a hyperbolic triangle is always less than 180°

In fact, given any angle α between 0° & 180°, there is a triangle with the sum of the interior angles equal to α: