CHAPTER 2: PLANE TRANSFORMATIONS

Section 2.1: Plane Symmetries

In this section we discuss transformations of the points of the plane.
Call this transformation f.
We can think of f as a rule that tells us where a point is moved.

The first transformation we examine is "rotation".

ex// Fix a point O in the plane & let f be the rotation around O (the "center of rotation") through 45° (the "angle of rotation")

So to define each **ROTATION** transformation we need

1. the centre of rotation, &
2. the angle of rotation.

A transformation is "rigid" if for every 2 points A & B in the plane, the distance between A & B is the same as the distance between the images f(A) & f(B).

Such transformations are called "symmetries".

There are 2 more plane symmetries that we will explore. g

The first is "translation". (a slide)

To define each TRANSLATION symmetry, we need ① it's vector (an oriented arrow)

(note: distances are preserved)

h

The second is "reflection". (a mirror)

To define each REFLECTION symmetry, we need ① its line of reflection ℓ.

Two symmetries f & g are considered equal if for every point A, $f(A) = g(A)$.

ex// Rotate the point P around O

a) by f, rotation of 90°

b) by g, rotation of −270°

(note: convention is that + angles go counterclockwise)

Other symmetries can be created by applying multiple symmetries at a time, ie, a "composition" of symmetries.

ex// Suppose t_1 is a translation along a vector of unit length in the upward direction & t_2 is the translation along a unit vector pointing rightward. Describe the symmetry obtained by composing t_1 & t_2.

The Classification Theorem for Plane Symmetries:

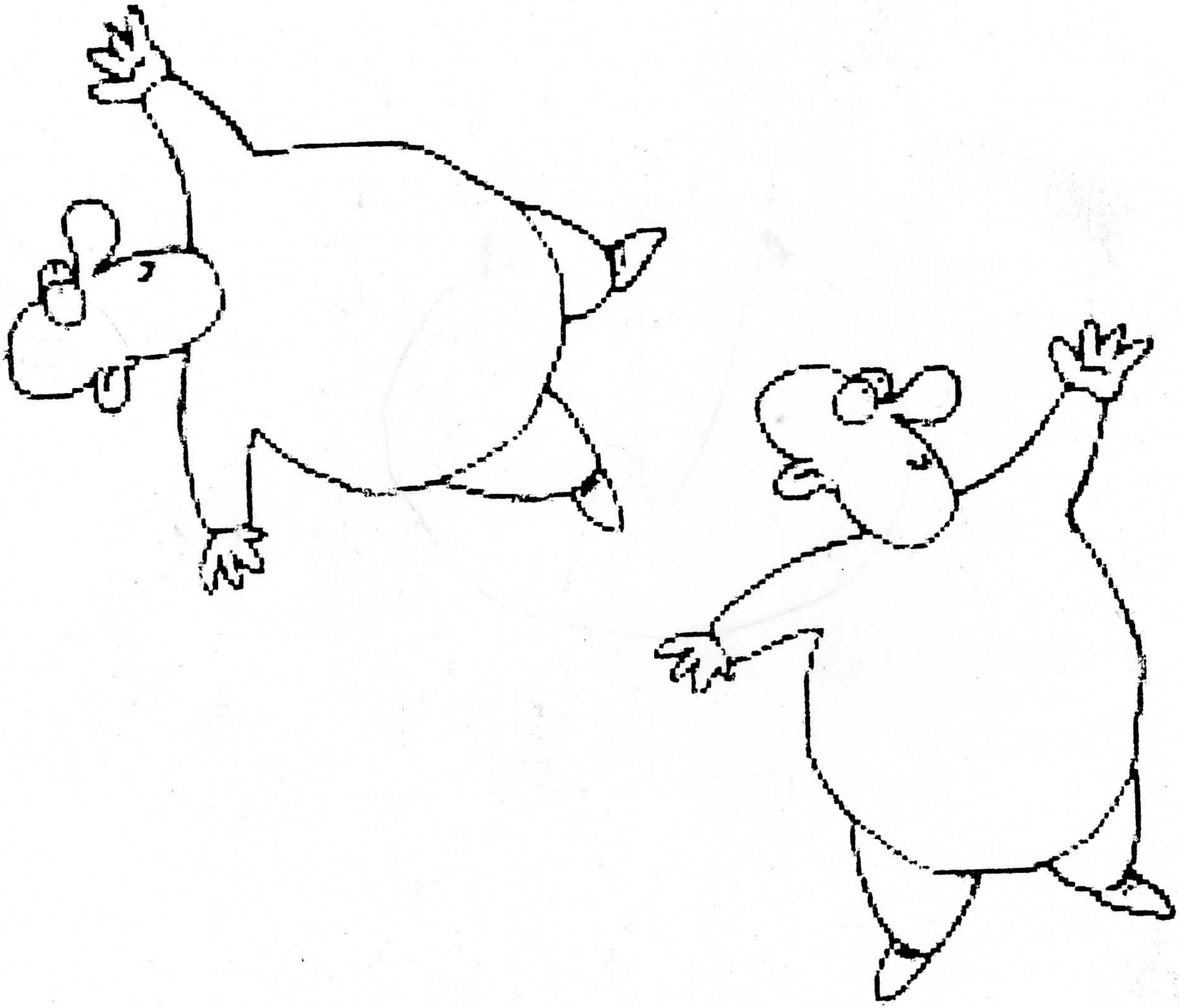
Every symmetry of the plane is either a composition of a rotation followed by a translation, or a reflection followed by a translation.

There is also one special symmetry called the "identity" symmetry that doesn't move any points in the plane.

It is either rotation through the angle of $0°$ ($=360°$) or translation through the 0 vector.
So the composition any symmetry f followed by identity symmetry is f itself.

Reflections followed by translations in the direction parallel to the line of reflection are "*glide reflections*".

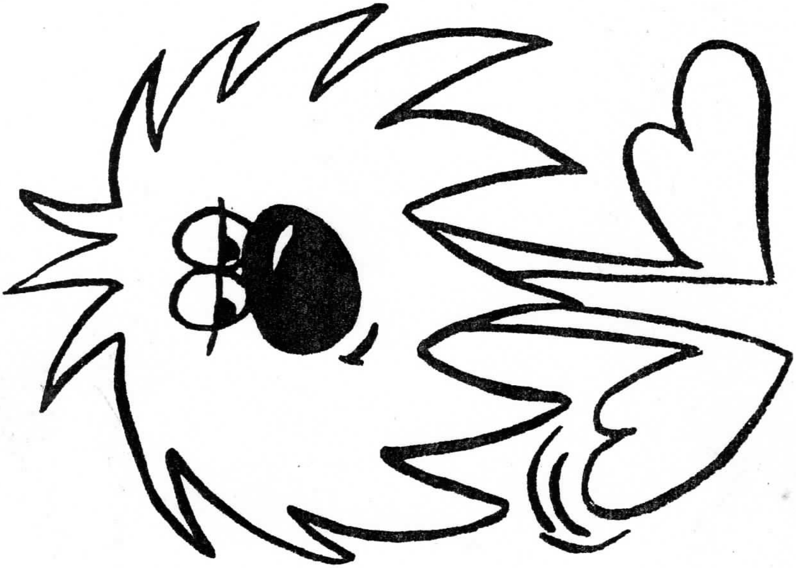
ex// Move the points A & B according to the <u>glide reflection f</u> that is a reflection over ℓ followed by a translation over $v = \uparrow\}2$



? What if we start with A & f(A) & are asked to identify the symmetry(ies) ourselves?

ex// One of the two characters is obtained from the other by rotating it around a point O & through a negative angle of rotation. Find the angle & center of rotation.

1. Identify the original & the image
2. Pick any 2 points on the original (easy to recognize) & their images
3. Connect B & f(B), C & f(C)
4. Bisect the lines
5. The intersection of the bisectors is the center of rotation.
6. connect a point (& its image) to the center of rot.
   - this is the angle of rotation



If we are given 2 points (or ~~images~~ objects) in the plane & we want to construct the line of reflection ...

① Choose a point A & identify its image

② Connect the point & its image

③ Bisect that line

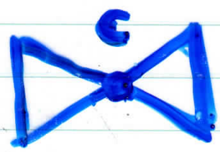
So now we can identify the symmetries applied given a point (object) & its image, or we can identify a point's (object's) image given a symmetry or a composition of symmetries to apply.

But these are not our typical ideas of what we are dealing with when we hear the word "symmetry".

We see this in the next section ...

Section 2.2: Symmetries of Plane Objects

A symmetry rearranges points in the plane. Some of the points may be moved outside the object, but some may end up within the object itself.

ex// 

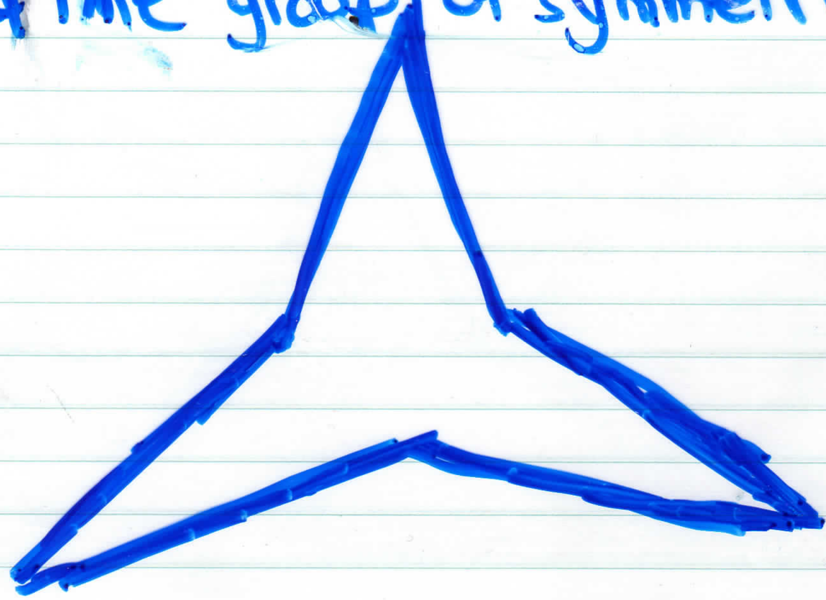
a) translate by $\vec{v}$

b) rotate $180°$ around c

Given an object O in the plane, the set of all symmetries of the plane that keep all of the points of O within O is the "group of symmetries" of O.

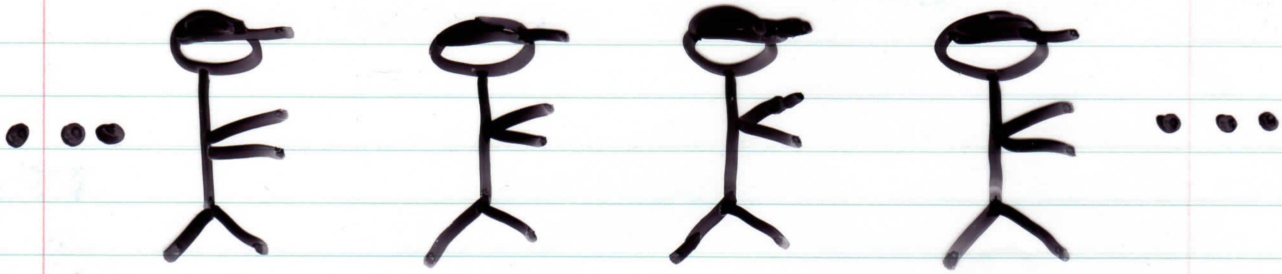
The elements of that set are the "symmetries of Ω".

ex// Find the group of symmetries of:



Section 2.3: Frieze Patterns

Lets look at an object that extends unboundedly on both sides



All translational symmetries have the following 2 properties:

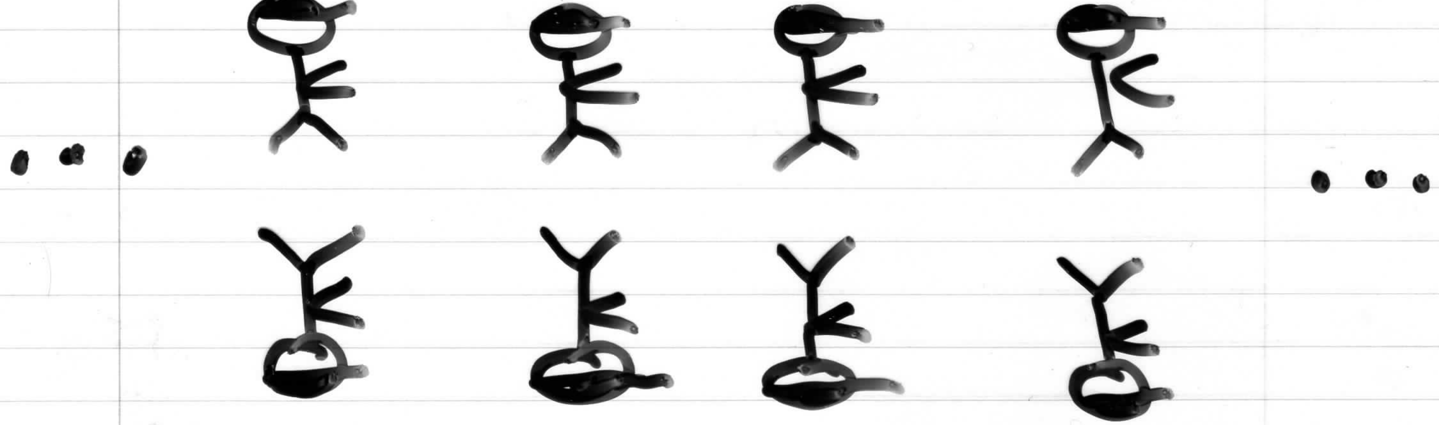
a) All of the translational symmetries are along vectors that are parallel to each other (possibly in opposite directions).

b) The sizes of the translational vectors range through all integer multiples of the smallest non-zero vector.

Objects in the plane satisfying these two properties are called "frieze patterns" & their groups of symmetries are called "frieze groups".

There are 7 classifications of frieze patterns. The one we just saw is F_1.

F_2 is the frieze pattern of reflection with respect to the horizonal line



Is that the only symmettry here?

We will only consider these two frieze patterns. The remaining 5 are covered in the text but will not be tested - cover as you wish.

Chapter 5: FRACTALS

Section 5.1: Some Basic Fractals

A fractal is an object that possesses self-similarity ie, part of the object looks like the entire thing.

Eventually the length of the smallest branches is smaller than their thickness, so we can't detect any more iterations (steps) with our naked eye. So after applying the branching operation *infinitely* many times, we get a "fractal tree".

So the method for making fractals is to start with an object & reapeatedly apply a chosen transformation.

? How can we make the "nicest" fractal tree?

What should we make the ratio of branches to trunk so the fractal has branches that do not become entangled, but just touch?

Another tree fractal:

1. Again start with a vertical line, but now draw new branches ($\frac{1}{3}$ as long) at $\frac{1}{3}$ & $\frac{2}{3}$ of the way up the trunk.

2. Repeat on each line segment...

We can do the same in 3D...

This tree does not look natural, why?

Because nature involves random & un-deterministic growth.

So if we choose the angles between branches randomly...

Section 5.2: More Basic Fractals

Not all fractals are "trees". We just start with any geometric shape & iterate a chosen transformation!

(note: the real fractal doesn't happen until we have iterated infinitely many times.

ex// The first 2 iterations are given below, draw the object obtained by iterating one more step.