Structure of the quotient algebra of compact-by-approximable operators for Banach spaces failing the approximation property

Hans-Olav Tylli

University of Helsinki
(joint with Henrik Wirzenius)

Banach Algebras 2019, Winnipeg
July 15, 2019
Basic references

Let $X, Y$ be Banach spaces, and

$$\mathcal{L}(X, Y) = \{\text{bounded linear operators } X \to Y\}$$

$$\mathcal{K}(X, Y) = \{\text{compact operators } X \to Y\}$$

$$\mathcal{A}(X, Y) = \{\text{approximable operators } X \to Y\} = \mathcal{F}(X, Y),$$

where $\mathcal{F}(X, Y) = \{\text{bounded finite rank operators } X \to Y\}$.

For $X = Y$ write $\mathcal{K}(X) := \mathcal{K}(X, X)$, $\mathcal{A}(X) := \mathcal{A}(X, X)$ etc.

$\mathcal{K}(X)$ and $\mathcal{A}(X)$ are closed two-sided ideals of $\mathcal{L}(X)$ in the operator norm, so $\mathcal{A}_X := \mathcal{K}(X)/\mathcal{A}(X)$ is a Banach algebra (the "compact-by-approximable" quotient algebra) with

$$\| T + \mathcal{A}(X) \| = \text{dist}(T, \mathcal{A}(X)) = \inf_{A \in \mathcal{A}(X)} \| T - A \|$$

whenever $X$ is a real or complex Banach space.
Setting

* Let $X$, $Y$ be Banach spaces, and

\[
\mathcal{L}(X, Y) = \{\text{bounded linear operators } X \to Y\}
\]

\[
\mathcal{K}(X, Y) = \{\text{compact operators } X \to Y\}
\]

\[
\mathcal{A}(X, Y) = \{\text{approximable operators } X \to Y\} = \mathcal{F}(X, Y),
\]

where $\mathcal{F}(X, Y) = \{\text{bounded finite rank operators } X \to Y\}$.

* For $X = Y$ write $\mathcal{K}(X) := \mathcal{K}(X, X)$, $\mathcal{A}(X) := \mathcal{A}(X, X)$ etc.

* $\mathcal{K}(X)$ and $\mathcal{A}(X)$ are closed two-sided ideals of $\mathcal{L}(X)$ in the operator norm, so $\mathfrak{A}_X := \mathcal{K}(X)/\mathcal{A}(X)$ is a Banach algebra (the "compact-by-approximable" quotient algebra) with

\[
\| T + \mathcal{A}(X) \| = \text{dist}(T, \mathcal{A}(X)) = \inf_{A \in \mathcal{A}(X)} \| T - A \|
\]

whenever $X$ is a real or complex Banach space.
\* \( \mathfrak{A}_X = \mathcal{K}(X)/\mathcal{A}(X) \) is a (non-unital) radical Banach algebra. (For real scalars radicality is here interpreted as

\[ \sigma_{\mathbb{R}}(T + \mathcal{A}(X)) = \{0\}, \quad T \in \mathcal{K}(X), \]

which follows from classical Riesz-Fredholm theory). The structure of \( \mathfrak{A}_X \) is very poorly understood, in part because of connection to approximation properties.

**Recall:** Banach space \( X \) has the *approximation property* (AP) if for all \( \varepsilon > 0 \) and all compact subsets \( K \subset X \) there is a finite rank operator \( T \in \mathcal{F}(X) \) such that

\[ \sup_{x \in K} \|x - T x\| < \varepsilon \]

- All classical Banach spaces have AP: spaces with a Schauder basis, \( L^p \)-spaces, \( C(K) \)-spaces etc.
- Enflo (1973): there is a Banach space \( X \subset c_0 \) without AP.
- Davie (1973) & Szankowski (1978): for \( 1 \leq p < \infty, p \neq 2 \), there is a closed subspace \( X \subset \ell^p \) failing AP.
Approximation properties 1

\[ \mathfrak{A}_X = \mathcal{K}(X)/\mathcal{A}(X) \] is a (non-unital) radical Banach algebra. (For real scalars radicality is here interpreted as

\[ \sigma_{\mathbb{R}}(T + \mathcal{A}(X)) = \{0\}, \quad T \in \mathcal{K}(X), \]

which follows from classical Riesz-Fredholm theory). The structure of \( \mathfrak{A}_X \) is very poorly understood, in part because of connection to approximation properties.

Recall: Banach space \( X \) has the approximation property (AP) if for all \( \varepsilon > 0 \) and all compact subsets \( K \subset X \) there is a finite rank operator \( T \in \mathcal{F}(X) \) such that

\[
\sup_{x \in K} \|x - Tx\| < \varepsilon
\]

- All classical Banach spaces have AP: spaces with a Schauder basis, \( L^p \)-spaces, \( C(K) \)-spaces etc.
- Enflo (1973): there is a Banach space \( X \subset c_0 \) without AP. Davie (1973) & Szankowski (1978): for \( 1 \leq p < \infty, p \neq 2 \), there is a closed subspace \( X \subset \ell^p \) failing AP.
Approximation properties 2

- $\mathcal{L}(\ell^2)$ does not have AP (Szankowski; 1981), and neither does the Calkin algebra $\mathcal{L}(\ell^2)/\mathcal{K}(\ell^2)$ (Godefroy & Saphar; 1989).
- Relevance for quotient algebra $\mathcal{A}_X$: if $X$ has AP, then $\mathcal{K}(X) = \mathcal{A}(X)$, so $\mathcal{A}_X = \{0\}$.
- The converse is a notorious open question: if $\mathcal{K}(X) = \mathcal{A}(X)$, does $X$ have AP? (Lindenstrauss & Tzafriri, 1977).
- Dales (2013): collection of results and open questions on $\mathcal{A}_X$.
  - Is there a $X$ without AP such that $0 < \dim(\mathcal{A}_X) < \infty$? (Still open)
  - Is there a $X$ such that $\mathcal{A}_X$ contains infinitely many non-trivial closed ideals? (Still open)
  - Further questions about factorisation, amenability of $\mathcal{K}(X)$ etc.
\[ \mathcal{L}(\ell^2) \] does not have AP (Szankowski; 1981), and neither does the Calkin algebra \( \mathcal{L}(\ell^2)/\mathcal{K}(\ell^2) \) (Godefroy & Saphar; 1989)

\* Relevance for quotient algebra \( \mathcal{A}_X \): if \( X \) has AP, then \( \mathcal{K}(X) = \mathcal{A}(X) \), so \( \mathcal{A}_X = \{0\} \).

\* The converse is a notorious open question: if \( \mathcal{K}(X) = \mathcal{A}(X) \), does \( X \) have AP? (Lindenstrauss & Tzafriri, 1977)

\* Dales (2013): collection of results and open questions on \( \mathcal{A}_X \).

- Is there a \( X \) without AP such that \( 0 < \dim(\mathcal{A}_X) < \infty \)? (Still open)

- Is there a \( X \) such that \( \mathcal{A}_X \) contains infinitely many non-trivial closed ideals? (Still open)

- Further questions about factorisation, amenability of \( \mathcal{K}(X) \) etc.
Warm-up: examples where \( \mathfrak{A}_X \neq \{0\} \)

⋆ Characterisation of the AP (Grothendieck): \( X \) has AP if and only if \( \mathcal{K}(Y, X) = \mathcal{A}(Y, X) \) for every Banach space \( Y \).

This easily provides examples of Banach spaces \( Z \) with a non-trivial quotient algebra \( \mathfrak{A}_Z \):

Example Suppose \( X \) is a Banach space failing the AP, and pick \( Y \) and \( T \in \mathcal{K}(Y, X) \setminus \mathcal{A}(Y, X) \). Let \( Z = Y \oplus X \) and define

\[
\tilde{T} : Z \to Z, \quad \tilde{T}(y, x) = (0, Ty).
\]

Then \( \tilde{T} \in \mathcal{K}(Z) \setminus \mathcal{A}(Z) \) and \( \mathfrak{A}_Z \neq \{0\} \).

⋆ Drawback: above \( Y = (Y_0, |\cdot|) \), where \( Y_0 \subseteq X \) linear subspace and \( |\cdot| \) complete norm in \( Y_0 \). (Hence \( Y \) preserves little of \( X \).)

⋆ Bachelis [B76]: Suppose that \( E \) is a Banach space having the bounded AP, \( E \oplus E \approx E \) and \( E \) contains a closed subspace \( X \) failing the AP. Then there is a closed subspace \( Z \subset E \) such that \( \mathfrak{A}_Z \neq \{0\} \). (This applies e.g. to \( E = \ell^p \) for \( 1 \leq p < \infty \) and \( p \neq 2 \), or \( E = c_0 \). Generalises Alexander (1974) for \( 2 < p < \infty \).)
Warm-up: examples where $\mathfrak{A}_X \neq \{0\}$

⋆ Characterisation of the AP (Grothendieck): $X$ has AP if and only if $\mathcal{K}(Y, X) = \mathcal{A}(Y, X)$ for every Banach space $Y$.

This easily provides examples of Banach spaces $Z$ with a non-trivial quotient algebra $\mathfrak{A}_Z$:

Example Suppose $X$ is a Banach space failing the AP, and pick $Y$ and $T \in \mathcal{K}(Y, X) \setminus \mathcal{A}(Y, X)$. Let $Z = Y \oplus X$ and define

$$\tilde{T} : Z \to Z, \tilde{T}(y, x) = (0, Ty).$$

Then $\tilde{T} \in \mathcal{K}(Z) \setminus \mathcal{A}(Z)$ and $\mathfrak{A}_Z \neq \{0\}$.

⋆ Drawback: above $Y = (Y_0, |\cdot|)$, where $Y_0 \subset X$ linear subspace and $|\cdot|$ complete norm in $Y_0$. (Hence $Y$ preserves little of $X$.)

⋆ Bachelis [B76]: Suppose that $E$ is a Banach space having the bounded AP, $E \oplus E \approx E$ and $E$ contains a closed subspace $X$ failing the AP. Then there is a closed subspace $Z \subset E$ such that $\mathfrak{A}_Z \neq \{0\}$. (This applies e.g. to $E = \ell^p$ for $1 \leq p < \infty$ and $p \neq 2$, or $E = c_0$. Generalises Alexander (1974) for $2 < p < \infty$.)
Examples with large $\mathcal{X}$, 1

Motivation: $c_0$ embeds isometrically into $\mathcal{K}(\ell^p) = \mathcal{A}(\ell^p)$ for $1 \leq p < \infty$ by the map $a \mapsto T_a$ for $a = (a_k) \in c_0$, where

$$T_a(x_k) = (a_kx_k), \quad (x_k) \in \ell^p.$$ 

Infinite-dimensional analogue for $\mathcal{X}$?

Proposition [TW19] Let $X_1, X_2, \ldots$ be sequence of Banach spaces with $\mathcal{A}_{X_n} \neq \{0\}$ for all $n \in \mathbb{N}$. Then there is a linear isomorphic embedding $c_0 \to \mathcal{X}$, where $X = (\bigoplus_{n \in \mathbb{N}} X_n)_p$ for $1 \leq p \leq \infty$. (Interpret as direct $c_0$-sum for $p = \infty$.)

Proof (idea): For $n \in \mathbb{N}$, pick $T_n \in \mathcal{K}(X_n) \setminus \mathcal{A}(X_n)$ with $\|T_n\| < 2$ and $\|T_n + \mathcal{A}(X_n)\| = 1$. Let $P_n : X \to X_n$, $(x_k) \mapsto x_n$ and $J_n : X_n \to X$, $x_n \mapsto (0, 0, \ldots, 0, x_n, 0, 0, \ldots)$ be the canonical maps. Then one may define a linear isomorphic embedding $c_0 \to \mathcal{X}$ by

$$\theta : (a_k) \mapsto \sum_{k=1}^{\infty} a_k J_k T_k P_k + \mathcal{A}(X).$$
Motivation: $c_0$ embeds isometrically into $\mathcal{K}(\ell^p) = \mathcal{A}(\ell^p)$ for $1 \leq p < \infty$ by the map $a \mapsto T_a$ for $a = (a_k) \in c_0$, where

$$T_a(x_k) = (a_k x_k), \quad (x_k) \in \ell^p.$$ 

Infinite-dimensional analogue for $\mathcal{A}_X$?

**Proposition [TW19]** Let $X_1, X_2, \ldots$ be sequence of Banach spaces with $\mathcal{A}_{X_n} \neq \{0\}$ for all $n \in \mathbb{N}$. Then there is a linear isomorphic embedding $c_0 \to \mathcal{A}_X$, where $X = (\bigoplus_{n \in \mathbb{N}} X_n)_p$ for $1 \leq p \leq \infty$. (Interpret as direct $c_0$-sum for $p = \infty$.)

**Proof (idea):** For $n \in \mathbb{N}$, pick $T_n \in \mathcal{K}(X_n) \setminus \mathcal{A}(X_n)$ with $\|T_n\| < 2$ and $\|T_n + \mathcal{A}(X_n)\| = 1$. Let $P_n : X \to X_n$, $(x_k) \mapsto x_n$ and $J_n : X_n \to X$, $x_n \mapsto (0, 0, \ldots, 0, x_n, 0, 0 \ldots)$ be the canonical maps. Then one may define a linear isomorphic embedding $c_0 \to \mathcal{A}_X$ by

$$\theta : (a_k) \mapsto \sum_{k=1}^{\infty} a_k J_k T_k P_k + \mathcal{A}(X).$$
Corollary. Let $1 < p < \infty$, $p \neq 2$. Then there is a closed subspace $X \subset \ell^p$ such that $\mathcal{A}_X$ contains a complemented copy of $c_0$.

Comment. ”Most” Banach spaces $Z$ contains a closed subspace isomorphic to $\ell^p$ (for some $1 \leq p < \infty$) or to $c_0$.

Proof (idea): By Bachelis [B76] there is a closed subspace $Y \subset \ell^p$ such that $\mathcal{A}_Y \neq \{0\}$. Put $X = (\oplus_{N} Y)_p \subset (\oplus_{N} \ell_p)_p \approx \ell^p$. By above Proposition, $\mathcal{A}_X$ contains a copy of $c_0$. Moreover, $\mathcal{A}_X$ is separable, so $c_0$ is complemented in $\mathcal{A}_X$ by Sobczyk’s theorem.

Here also fact: if $X^*$ is separable, then $K(X)$ (as well as $\mathcal{A}_X$) is separable. Reason: $K := (B_{X^*}, w^*) \times (B_{X^{**}}, w^*)$ is compact metrisable, and isometric embedding $\chi : K(X) \to C(K)$, where

$$\chi(T)(x^*, x^{**}) = \langle T^* x^*, x^{**} \rangle, \quad T \in K(X), \ (x^*, x^{**}) \in K.$$
Examples with large $\mathfrak{A}_X$, II

**Corollary.** Let $1 < p < \infty$, $p \neq 2$. Then there is a closed subspace $X \subset \ell^p$ such that $\mathfrak{A}_X$ contains a complemented copy of $c_0$.

**Comment.** "Most" Banach spaces $Z$ contains a closed subspace isomorphic to $\ell^p$ (for some $1 \leq p < \infty$) or to $c_0$.

**Proof** (idea): By Bachelis [B76] there is a closed subspace $Y \subset \ell^p$ such that $\mathfrak{A}_Y \neq \{0\}$. Put $X = (\bigoplus_N Y)_p \subset (\bigoplus_N \ell_p)_p \approx \ell^p$. By above Proposition, $\mathfrak{A}_X$ contains a copy of $c_0$. Moreover, $\mathfrak{A}_X$ is separable, so $c_0$ is complemented in $\mathfrak{A}_X$ by Sobczyk’s theorem.

Here also fact: *if $X^*$ is separable, then $K(X)$ (as well as $\mathfrak{A}_X$) is separable.* Reason: $K := (B_{X^*}, w^*) \times (B_{X^{**}}, w^*)$ is compact metrisable, and isometric embedding $\chi : K(X) \to C(K)$, where

$$\chi(T)(x^*, x^{**}) = \langle T^* x^*, x^{**} \rangle, \quad T \in \mathcal{K}(X), \ (x^*, x^{**}) \in K.$$
Question: Banach algebraic relevance of linear embedding $\theta$?

- Above $\theta: c_0 \to \mathcal{A}_X$ is never an algebra homomorphism (since $\mathcal{A}_X$ radical B-algebra, but $c_0$ not so)

- $\theta(c_0)$ is a commuting set, and generates a closed commutative subalgebra $\mathcal{A}[\theta(c_0)]$ of $\mathcal{A}_X$. But there is closed subspace $X \subset \ell^p$ for $p \neq 2$, so that $\mathcal{A}_X$ is non-commutative (whence $\mathcal{A}[\theta(c_0)] \neq \mathcal{A}_X$).

- In the Corollary one may choose the embedding $\theta$ so that
  (i) $\theta(a) \cdot \theta(b) = 0$ for all $a, b \in c_0$,
  alternatively
  (ii) for each $n \in \mathbb{N}$ there are $a^{(k)} \in c_0$ for $k = 1, \ldots, n$ such that $\theta(a^{(1)}) \cdots \theta(a^{(n)}) \neq 0$. (This means that $\mathcal{A}[\theta(c_0)]$, as well as $\mathcal{A}_X$, is not nilpotent. Part (ii) is based on [B76].)
Question: Banach algebraic relevance of linear embedding $\theta$?

- Above $\theta : c_0 \to A_X$ is never an algebra homomorphism (since $A_X$ radical B-algebra, but $c_0$ not so)
- $\theta(c_0)$ is a commuting set, and generates a closed commutative subalgebra $A[\theta(c_0)]$ of $A_X$. But there is closed subspace $X \subset \ell^p$ for $p \neq 2$, so that $A_X$ is non-commutative (whence $A[\theta(c_0)] \neq A_X$).
- In the Corollary one may choose the embedding $\theta$ so that
  (i) $\theta(a) \cdot \theta(b) = 0$ for all $a, b \in c_0$,
  alternatively
  (ii) for each $n \in \mathbb{N}$ there are $a^{(k)} \in c_0$ for $k = 1, \ldots, n$ such that $\theta(a^{(1)}) \cdots \theta(a^{(n)}) \neq 0$. (This means that $A[\theta(c_0)]$, as well as $A_X$, is not nilpotent. Part (ii) is based on [B76].)
Recall: $X$ has the **compact approximation property** (CAP) if for all $\varepsilon > 0$ and all compact subsets $K \subset X$ there is a compact operator $T \in \mathcal{K}(X)$ such that

$$\sup_{x \in K} ||x - Tx|| < \varepsilon$$

Moreover, $X$ has the **bounded compact approximation property** (BCAP) if there is constant $M < \infty$ such that $T \in \mathcal{K}(X)$ above can always be chosen with $||T|| \leq M$.

This points to a relevant class of spaces!

**Proposition** (Dales, [TW19]) If $X$ has BCAP but fails AP, then $\dim(\mathfrak{A}_X) = \infty$. Moreover, there is $T \in \mathcal{K}(X)$ such that $T^n \notin \mathcal{A}(X)$ for every $n \in \mathbb{N}$.

(Recall that $||T^n + \mathcal{A}(X)||^{1/n} \to 0$ as $n \to \infty$ by radicality of $\mathfrak{A}_X$.)
Compact approximation property

Recall: $X$ has the **compact approximation property** (CAP) if for all $\varepsilon > 0$ and all compact subsets $K \subset X$ there is a compact operator $T \in \mathcal{K}(X)$ such that

$$
\sup_{x \in K} \|x - Tx\| < \varepsilon
$$

Moreover, $X$ has the **bounded compact approximation property** (BCAP) if there is constant $M < \infty$ such that $T \in \mathcal{K}(X)$ above can always be chosen with $\|T\| \leq M$.

This points to a relevant class of spaces!

**Proposition** (Dales, [TW19]) If $X$ has BCAP but fails AP, then $\dim(\mathfrak{A}_X) = \infty$. Moreover, there is $T \in \mathcal{K}(X)$ such that $T^n \notin \mathcal{A}(X)$ for every $n \in \mathbb{N}$.

(Recall that $\|T^n + \mathcal{A}(X)\|^{1/n} \to 0$ as $n \to \infty$ by radicality of $\mathfrak{A}_X$.)
Spaces with BCAP that fail AP

**Question:** Do such Banach spaces $X$ exist?

**Theorem.** (Willis [W92]) There are Banach spaces $Z$ having the BCAP (with constant $M = 1$) that fail AP.

**Outline of the construction:**

Let $X$ be a Banach space failing AP. Then there is $c > 0$ and a compact $K = \overline{\text{conv}}\{x_n \mid n \in \mathbb{N}\} \subset X$, where $\|x_n\| \to 0$, such that

$$\sup_{x \in K} \|Tx - x\| \geq c \quad \text{for all } T \in \mathcal{F}(X).$$

For all $0 < s < 1$, let $U_s = \overline{\text{absconv}}\{x_n/\|x_n\|^s \mid n \in \mathbb{N}\}$ and let $Y_s = \text{span } U_s$ endowed with the complete norm

$$|y|_s = \inf\{\lambda > 0 \mid y \in \lambda U_s\}.$$ 

Let $Z = \text{span}\{y\chi_{(s,t)} \mid 0 < s < t < 1, y \in Y_s\}$ with the norm

$$\|f\| = \int_0^1 |f(s)|_s ds.$$ 

Main step: the completion $Z$ of $Z$ has BCAP but fails AP.
Spaces with BCAP that fail AP

**Question**: Do such Banach spaces $X$ exist?

**Theorem.** (Willis [W92]) There are Banach spaces $Z$ having the BCAP (with constant $M = 1$) that fail AP.

Outline of the construction:

Let $X$ be a Banach space failing AP. Then there is $c > 0$ and a compact $K = \text{conv}\{x_n \mid n \in \mathbb{N}\} \subset X$, where $\|x_n\| \to 0$, such that

$$\sup_{x \in K} \|Tx - x\| \geq c \quad \text{for all } T \in \mathcal{F}(X).$$

For all $0 < s < 1$, let $U_s = \text{absconv}\left\{\frac{x_n}{\|x_n\|s} \mid n \in \mathbb{N}\right\}$ and let $Y_s = \text{span } U_s$ endowed with the complete norm

$$|y|_s = \inf\{\lambda > 0 \mid y \in \lambda U_s\}.$$

Let $Z = \text{span}\{y\chi(s,t) \mid 0 < s < t < 1, y \in Y_s\}$ with the norm

$$\|f\| = \int_0^1 |f(s)|_s ds.$$ Main step: the completion $Z$ of $Z$ has BCAP but fails AP.
Theorem. [TW19] Let \( Z \) be a Willis space. Then there is a linear isomorphic embedding \( \theta : c_0 \to A_Z \).

Idea of proof: Fix \( 0 < s < t < 1 \). The following diagram commutes up to a constant (which depends on \( s, t \)).

Here \( R_s \notin A(Y_s, X) \) and \( T_s \in K(Y_s, Z) \). Thus \( P_0 T_s \in K(Y_s, Z_0) \setminus A(Y_s, Z_0) \).

Hence the closed complemented subspace \( Z_0 \) has (B)CAP but fails AP. Consequently \( A_{Z_0} \neq \{0\} \).
Theorem. [TW19] Let $Z$ be a Willis space. Then there is a linear isomorphic embedding $\theta : c_0 \to \mathcal{A}_Z$.

Idea of proof: Fix $0 < s < t < 1$. The following diagram commutes up to a constant (which depends on $s$, $t$).

Here $R_s \notin \mathcal{A}(Y_s, X)$ and $T_s \in \mathcal{K}(Y_s, Z)$. Thus $P_0 T_s \in \mathcal{K}(Y_s, Z_0) \setminus \mathcal{A}(Y_s, Z_0)$.

Hence the closed complemented subspace $Z_0$ has (B)CAP but fails AP. Consequently $\mathcal{A}_{Z_0} \neq \{0\}$. 

Theorem. [TW19] Let $Z$ be a Willis space. Then there is a linear isomorphic embedding $\theta : c_0 \to \mathcal{A}_Z$.

Idea of proof: Fix $0 < s < t < 1$. The following diagram commutes up to a constant (which depends on $s$, $t$).

Here $R_s \notin \mathcal{A}(Y_s, X)$ and $T_s \in \mathcal{K}(Y_s, Z)$. Thus $P_0 T_s \in \mathcal{K}(Y_s, Z_0) \setminus \mathcal{A}(Y_s, Z_0)$.

Hence the closed complemented subspace $Z_0$ has (B)CAP but fails AP. Consequently $\mathcal{A}_{Z_0} \neq \{0\}$.
Fix interlacing sequences \((s_n)\) and \((t_n)\) of \((0, 1)\) such that 
\[ s_n < t_n < s_{n+1} \]
for all \(n\) and \(s_n \to 1\) as \(n \to \infty\). Let \(Z_n \subset Z\) be the closed subspace consisting of functions in \(Z\) supported in 
\((s_n, t_n) \subset (0, 1)\). By above argument \(A_{Z_n} \neq \{0\}\) for all \(n \in \mathbb{N}\). By the general Proposition \(A_{\bigoplus_n Z_n 1}\) contains an isomorphic copy of \(c_0\). Deduce that \(c_0\) also embeds into \(A_Z\) by the following fact.

**Proposition.** Let \(X, Y\) be Banach spaces and suppose that 
\(A_X \neq \{0\}\). Then \(A_X\) is algebra isomorphic to a subalgebra of 
\(A_{X \oplus Y}\).

**Comment:** Theorem holds also for related separable reflexive Willis spaces \(Z_p\) with \(1 < p < \infty\) and \(p \neq 2\), where \(Z_p\) quotient of closed subspace of \(L^p\).
Fix interlacing sequences \((s_n)\) and \((t_n)\) of \((0, 1)\) such that 
\(s_n < t_n < s_{n+1}\) for all \(n\) and \(s_n \to 1\) as \(n \to \infty\). Let \(Z_n \subset Z\) be the closed subspace consisting of functions in \(Z\) supported in 
\((s_n, t_n) \subset (0, 1)\). By above argument \(\mathcal{A}_{Z_n} \neq \{0\}\) for all \(n \in \mathbb{N}\). By 
the general Proposition \(\mathcal{A}_{(\oplus_n Z_n)_1}\) contains an isomorphic copy of \(c_0\). Deduce that \(c_0\) also embeds into \(\mathcal{A}_Z\) by the following fact.

**Proposition.** Let \(X, Y\) be Banach spaces and suppose that 
\(\mathcal{A}_X \neq \{0\}\). Then \(\mathcal{A}_X\) is algebra isomorphic to a subalgebra of 
\(\mathcal{A}_X \oplus Y\).

**Comment:** Theorem holds also for related separable reflexive Willis 
spaces \(Z_p\) with \(1 < p < \infty\) and \(p \neq 2\), where \(Z_p\) quotient of closed 
subspace of \(L^p\).
Further big quotient algebras

Let \((G_m)\) be sequence of finite dimensional spaces dense in the Banach-Mazur distance in the class of finite dimensional Banach spaces. Let \(C_p = (\bigoplus_{m\in\mathbb{N}} G_m)_{\ell^p}\) for \(1 \leq p < \infty\). \(C_1\) is Johnson’s complementably universal conjugate space.

Proposition. [TW19] \(\mathcal{A}_{C_1^*}\) contains a linear isomorphic copy of \(c_0\) (but \(\mathcal{A}_{C_1} = \{0\}\) since \(C_1\) has AP).

Johnson (1972) & Figiel (1973): every compact \(T \in \mathcal{K}(X, Y)\) factors compactly through a subspace of \(C_p\), that is, there is a closed subspace \(M \subset C_p\) and compact operators \(A \in \mathcal{K}(X, M), B \in \mathcal{K}(M, Y)\) so that \(T = B \circ A\). Let \(1 \leq p < \infty\) and

\[
\mathcal{I} = \{ W \subset C_p \mid W \text{ closed infinite dimensional subspace}\}.
\]

Consider \(Z_{FJ}^p = (\bigoplus_{W \in \mathcal{I}} W)_{\ell^p}\), which is a universal compact factorisation space.

Proposition. [TW19] There is an uncountable index set \(\Gamma\) such that \(\mathcal{A}_{Z_{FJ}^p}\) contains a isomorphic copy of non-separable space \(c_0(\Gamma)\).
Further big quotient algebras

Let \((G_m)\) be sequence of finite dimensional spaces dense in the Banach-Mazur distance in the class of finite dimensional Banach spaces. Let \(C_p = (\bigoplus_{m \in \mathbb{N}} G_m)_{\ell^p}\) for \(1 \leq p < \infty\). \(C_1\) is Johnson’s complementably universal conjugate space.

Proposition. [TW19] \(A_{C_1^*}\) contains a linear isomorphic copy of \(c_0\) (but \(A_{C_1} = \{0\}\) since \(C_1\) has AP).

Johnson (1972) & Figiel (1973): every compact \(T \in \mathcal{K}(X, Y)\) factors compactly through a subspace of \(C_p\), that is, there is a closed subspace \(M \subset C_p\) and compact operators \(A \in \mathcal{K}(X, M), B \in \mathcal{K}(M, Y)\) so that \(T = B \circ A\). Let \(1 \leq p < \infty\) and

\[
\mathcal{I} = \{ W \subset C_p \mid W \text{ closed infinite dimensional subspace} \}.
\]

Consider \(Z_{FJ}^p = (\bigoplus_{W \in \mathcal{I}} W)_{\ell^p}\), which is a universal compact factorisation space.

Proposition. [TW19] There is an uncountable index set \(\Gamma\) such that \(A_{Z_{FJ}^p}\) contains a isomorphic copy of non-separable space \(c_0(\Gamma)\).
Concluding remarks and problems

Conclusions: size of $A(X) \subset K(X)$ is very subtle for operators $X \to X$, where $X$ is given Banach space failing AP.

Problems: (i) how embed other spaces $Z$ into $A_X$?
(ii) how preserve Banach algebra properties?

For direct sums the following variant becomes relevant:
(iii) Given $X, Y$ such that $X^*$ and $Y$ both fail the AP, is it always true that

$$A(X, Y) \subsetneq K(X, Y)?$$

Somewhat surprisingly this is not so! For example, if $X$ has type 2 and $Y$ has cotype 2, then $K(X, Y) = A(X, Y)$ (Kwapień, Maurey, John).

This matters e.g. for structure of closed ideals of $A_X$ (Cf. next talk by Henrik Wirzenius.)

THANKS!
Concluding remarks and problems

**Conclusions:** size of $\mathcal{A}(X) \subset \mathcal{K}(X)$ is very subtle for operators $X \rightarrow X$, where $X$ is given Banach space failing AP.

**Problems:** (i) how embed other spaces $Z$ into $\mathcal{A}_X$? 
(ii) how preserve Banach algebra properties?

For direct sums the following variant becomes relevant:

(iii) Given $X$, $Y$ such that $X^*$ and $Y$ both fail the AP, is it always true that

$$\mathcal{A}(X, Y) \nsubseteq \mathcal{K}(X, Y)?$$

Somewhat surprisingly this is not so! For example, if $X$ has type 2 and $Y$ has cotype 2, then $\mathcal{K}(X, Y) = \mathcal{A}(X, Y)$ (Kwapień, Maurey, John).

This matters e.g. for structure of closed ideals of $\mathcal{A}_X$ (Cf. next talk by Henrik Wirzenius.)

**THANKS!**