Operator-valued Schur multipliers and Herz-Schur multipliers of dynamical systems

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Joint work with
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Outline of the Talk

- Measurable Schur multipliers
- Herz-Schur multipliers
- Schur A-multipliers
- Herz-Schur multipliers for crossed products and transference
- Classes of Herz-Schur multipliers: central and convolution multipliers.
- Approximation
BACKGROUND: MEASURABLE SCHUR MULTIPLIERS

$(X, \mu)$ and $(Y, \nu)$ standard measure spaces
$H_1 = L^2(X, \mu), H_2 = L^2(Y, \nu)$

$\mathcal{B}(H_1, H_2)$ and $\mathcal{K}(H_1, H_2)$ the spaces of all bounded and resp. compact linear operators from $H_1$ into $H_2$.

$L^2(Y \times X)$ identify with $S_2(H_1, H_2)$, the space of all Hilbert-Schmidt operators, via $k \mapsto T_k$,

$$(T_k f)(y) = \int k(y, x)f(x) d\mu(x).$$

For measurable complex-valued function $\varphi$ on $Y \times X$ define $S_{\varphi}$ on $S_2(H_1, H_2)$ by

$$S_{\varphi} : T_k \mapsto T_{\varphi k}.$$  

$\varphi$ is called a (measurable) Schur multiplier if

$$\|S_{\varphi}(T_k)\|_{\text{op}} \leq C\|T_k\|_{\text{op}}, \forall k \in L^2(Y \times X).$$
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$S_\varphi$ has then a unique weak* continuous extension to $B(H_1, H_2)$ denoted by $S_\varphi$. Let $\mathcal{S}(X, Y)$ the set of all Schur multipliers.

**Theorem (Haagerup, Peller)**

Let $\varphi \in L^\infty(X \times Y)$. TFAE

(i) $\varphi$ is a Schur multiplier;

(ii) there exist weakly measurable functions $a : X \to l^2$ and $b : Y \to l^2$ such that

\[ \varphi(x, y) = (a(x), b(y))_\varphi = b(y)^* a(x), \text{ a.e. on } X \times Y \]

and
\[ \operatorname{esssup}_{x \in X} \|a(x)\|_2 \operatorname{esssup}_{y \in Y} \|b(y)\|_2 < \infty. \]

The correspondence $\varphi \mapsto S_\varphi$ is a complete isometry from $\mathcal{S}(X, Y)$ to $CB_{L^\infty(X), L^\infty(Y)}(\mathcal{K}(L^2(X), L^2(Y)))$, completely bounded maps on $\mathcal{K}(L^2(X), L^2(Y))$ which are $L^\infty(X)$-$L^\infty(Y)$-bimodular.
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Let $G$ be a locally compact group, $\lambda : G \to \mathcal{B}(L^2(G))$ the left regular representation of $G$, $\lambda(s)\xi(x) = \xi(s^{-1}x)$.

$C^*_r(G) = \lambda(L^1(G))\|\cdot\|_\text{op}$ the reduced $C^*$-algebra of $G$:

$\VN(G) = C^*_r(G)^\text{WOT} \subset \mathcal{B}(L^2(G))$ the von Neumann algebra of $G$

$A(G) = \{ s \mapsto (\lambda(s)\xi, \eta) = \bar{\eta} \ast \check{\xi} : \xi, \eta \in L^2(G) \} \subset C_0(G)$

the Fourier algebra of $G$.

We have $\VN(G) = A(G)^*$: if $T \in \VN(G)$ and $u(s) = (\lambda(s)\xi, \eta)$,

$$\langle T, u \rangle = (T\xi, \eta).$$

**Definition**

A function $u : G \to \mathbb{C}$ is called a (completely bounded) multiplier of $A(G)$ if the map

$$m_u : v \mapsto uv$$

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A function $u : G \to \mathbb{C}$ is called a **(completely bounded) multiplier** of $A(G)$ if the map

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Let $MA(G)$ ($M^{cb}A(G)$) be the set of (completely bounded) multipliers of $A(G)$.

**Theorem (J de Canniere & U.Haagerup)**

Let $u : G \to \mathbb{C}$ be a bounded continuous function. The following are equivalent:

- $u \in M^{cb}A(G)$;
- There exists a (unique) bounded weak* continuous completely bounded map $T$ on $VN(G)$ such that $T(\lambda_s) = u(s)\lambda_s$;
- There exists a completely bounded linear map $R$ on $C^*_r(G)$ such that
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**Embedding into the Schur Multipliers**

Let $\mathcal{S}(G)$ be the measurable Schur multipliers w.r.t. the Haar measure. $\varphi \in \mathcal{S}(G)$ is called *invariant* ($\varphi \in \mathcal{S}_{\text{inv}}(G)$) if for each $r \in G$,

$$\varphi(s, t) = \varphi(sr, tr), \quad \text{a.e.}$$

Given $u : G \to \mathbb{C}$, let $N(u) : G \times G \to \mathbb{C}$ be the function given by

$$N(u)(s, t) = u(st^{-1}), \quad s, t \in G.$$

**Theorem (Gilbert ’80, Bożeiko-Fendler ’84, Jolissaint, ’92)**

$u \in M^{cb}A(G)$ iff there exists a Hilbert space $H$ and measurable $a, b : G \to H$, such that $N(u)(s, t) = \langle a(s), b(t) \rangle_H$ a.e. and

$$\text{esssup}_s \|a(s)\|_2 \text{ esssup}_t \|b(t)\|_2 < \infty.$$ i.e. $N(u) \in \mathcal{S}(G)$.

**Theorem (Spronk ’04, Neufang-Ruan-Spronk ’07, Alaghmandan, Todorov, & T ’17)**

Let $G$ be second countable or discrete. The map $N$ is a surjective complete isometry from $M^{cb}A(G)$ to $\mathcal{S}_{\text{inv}}(G)$. 
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Non-Schur multiplier: Transformer of triangular truncation
\[ \varphi(x, y) = \chi_\Delta(x, y), \text{ where } \Delta = \{(x, y) \in [0, 1]^2 : x \leq y\} \]
(Gohberg-Krein)

Let \( G \) be a locally compact abelian group with Haar measure \( m \) and
\[ \varphi(x, y) = f(x - y), \text{ where } f : G \to \mathbb{C}. \text{ Then } \varphi \text{ is a Schur multiplier (w.r.t. } m) \text{ iff } f = \hat{\mu}, \mu \in M(\hat{G}), \text{ where } M(\hat{G}) \text{ is the space of complex bounded measures on the dual group } \hat{G} \text{ (Bożejko)}\]
Why are we interested?

- generalization of entrywise product of matrices
- link to perturbation theory through double operator integrals
- $M_{cb}^A(G)$
- Herz-Schur multipliers give rise to completely bounded maps on $VN(G)$ and $C^*_r(G)$ so link properties of a group and its associated operator algebras...
- ...approximation
Schur $A$-multipliers

Let $(X, \mu)$, $(Y, \nu)$ be standard measure spaces. Let $A$ be a non-degenerate separable $C^*$-algebra, $A \subset \mathcal{B}(H)$.

For $k \in L^2(Y \times X, \mathcal{B}(H))$ let $T_k : L^2(X, H) \to L^2(Y, H)$ be given by

$$(T_k \xi)(y) = \int_X k(y, x) \xi(x) d\mu(x).$$

If $\mathcal{M} \subseteq \mathcal{B}(H)$ is a $C^*$-subalgebra, let

$$S_2(Y \times X, \mathcal{M}) = \{ T_k : k \in L^2(Y \times X, \mathcal{M}) \}.$$

Note that, if $w \in L^2(Y \times X)$ and $a \in \mathcal{M}$ then $T_{w \otimes a} = T_w \otimes a$. giving $S_2(Y \times X, \mathcal{M})$ is norm dense in $\mathcal{K}(L^2(X), L^2(Y)) \otimes \mathcal{M}$. 
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Let $\varphi : X \times Y \to CB(A, \mathcal{B}(H))$ be a weakly measurable essentially bounded function. For $k \in L^2(Y \times X, A)$, let $\varphi \cdot k : Y \times X \to \mathcal{B}(H)$ be the function given by

$$(\varphi \cdot k)(y, x) = \varphi(x, y)(k(y, x)), \quad (y, x) \in Y \times X.$$ 

Let

$$S_\varphi : S_2(Y \times X, A) \to S_2(Y \times X, \mathcal{B}(H))$$

be the linear map given by

$$S_\varphi(T_k) = T_{\varphi \cdot k}, \quad k \in L^2(Y \times X, A).$$

**Definition**

$\varphi : X \times Y \to CB(A, \mathcal{B}(H))$ will be called a **Schur $A$-multiplier** if the map $S_\varphi$ is completely bounded.
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**Definition**

$\varphi : X \times Y \to CB(A, \mathcal{B}(H))$ will be called a Schur $A$-multiplier if the map $S_{\varphi}$ is completely bounded.
Let \( \varphi : X \times Y \to CB(A, \mathcal{B}(H)) \) be a weakly measurable essentially bounded function. For \( k \in L^2(Y \times X, A) \), let \( \varphi \cdot k : Y \times X \to \mathcal{B}(H) \) be the function given by

\[
(\varphi \cdot k)(y, x) = \varphi(x, y)(k(y, x)), \quad (y, x) \in Y \times X.
\]

Let

\[
S_\varphi : S_2(Y \times X, A) \to S_2(Y \times X, \mathcal{B}(H))
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**Definition**

\( \varphi : X \times Y \to CB(A, \mathcal{B}(H)) \) will be called a \textit{Schur A-multiplier} if the map \( S_\varphi \) is completely bounded.
If ϕ is a Schur A-multiplier then the map $S_\varphi$ possesses a completely bounded extension to a map from $\mathcal{K} \otimes A$ into $\mathcal{K} \otimes \mathcal{B}(H)$.

Let $\mathcal{S}(X, Y; A)$ be the space of all Schur A-multipliers and endow it with the norm

$$\|\varphi\|_m = \|S_\varphi\|_{cb}.$$ 

- The correspondence $\varphi \mapsto S_\varphi$ from $\mathcal{S}(X, Y, A)$ to $\text{CB}_{L^\infty(X) \otimes I, L^\infty(Y) \otimes I}(\mathcal{K} \otimes A, \mathcal{K} \otimes \mathcal{B}(H))$ is a completely isometric isomorphism.
If φ is a Schur A-multiplier then the map $S_φ$ possesses a **completely bounded extension** to a map from $\mathcal{K} \otimes A$ into $\mathcal{K} \otimes \mathcal{B}(H)$.

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**Theorem**

Let $\varphi : X \times Y \to \text{CB}(A, \mathcal{B}(H))$ be a weakly measurable function. TFAE:

(i) $\varphi$ is a Schur $A$-multiplier;

(ii) there exist a Hilbert space $K$, a non-degenerate $*$-representation $\rho : A \to \mathcal{B}(K)$ and weakly measurable maps $V : X \to \mathcal{B}(H, K)$ and $W : Y \to \mathcal{B}(H, K)$ with

$$\text{esssup}_{x \in X} \|V(x)\| < \infty \quad \text{and} \quad \text{esssup}_{y \in Y} \|W(y)\| < \infty,$$

such that, for all $a \in A$,

$$\varphi(x, y)(a) = W^*(y)\rho(a)V(x),$$

for almost all $(x, y) \in X \times Y.$
HERZ-SCHUR MULTIPLIERS FOR CROSSED PRODUCTS

Let $A$ be a separable $C^*$-algebra, $A \subset \mathcal{B}(H)$, and $(A, G, \alpha)$ be a $C^*$-dynamical system ($\alpha : G \to \text{Aut}(A)$).

Let $L^1(G, A)$ be the space of all $A$-valued integrable functions on $G$; it has a structure of $*$-algebra.

Let $\pi : A \to \mathcal{B}(L^2(G, H))$ be the $*$-representation defined by

$$(\pi(a)\xi)(t) = \alpha_{t^{-1}}(a)(\xi(t)), \ t \in G,$$

and $\lambda : G \to \mathcal{B}(L^2(G, H))$ be the (continuous) unitary representation given by

$$(\lambda_t \xi)(s) = \xi(t^{-1}s), \ s, t \in G.$$

The pair $(\pi, \lambda)$ is a covariant representation of $(A, G, \alpha)$, i.e. $\lambda_t \pi(a) = \pi(\alpha_t(a)) \lambda_t, \ t \in G$, that gives rise to a $*$-representation $\pi \ltimes \lambda : L^1(G, A) \to \mathcal{B}(L^2(G, H))$ given by

$$(\pi \ltimes \lambda)(f) = \int_G \pi(f(s)) \lambda_s ds, \ f \in L^1(G, A).$$

The reduced crossed product $A \rtimes_{\alpha, r} G$ of $A$ by $\alpha$ is, by definition, the closure of $(\pi \ltimes \lambda)(L^1(G, A))$ in the operator norm of $\mathcal{B}(L^2(G, H))$. 

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HERZ-SCHUR MULTIPLIERS FOR CROSSED PRODUCTS

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Definition

A pointwise measurable function $F : G \to CB(A)$ will be called a *Herz-Schur $(A, G, \alpha)$-multiplier* if the map

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given by

$$S_F((\pi \otimes \lambda)(f)) = (\pi \otimes \lambda)(F \cdot f)$$

is completely bounded, where $(F \cdot f)(s) = F(s)(f(s)), f \in L^1(G, A)$.

Let $\mathcal{G}(A, G, \alpha)$ be the set of all Herz-Schur $(A, G, \alpha)$-multipliers.

If $F \in \mathcal{G}(A, G, \alpha)$ then $S_F$ extends to a completely bounded map $S_F$ on $A \rtimes_{r, \alpha} G$. We let $\|F\|_m = \|S_F\|_{cb}$.

*If $G$ is discrete, Herz-Schur multipliers of dynamical systems were also introduced and studied by Bedos and Conti ’15.*
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**Transference Result**

For $F : G \to CB(A)$, let $\mathcal{N}(F) : G \times G \to CB(A)$ be given by

$$\mathcal{N}(F)(s, t)(a) = \alpha_{t^{-1}}(F(ts^{-1})(\alpha_t(a))), \quad a \in A, \; s, t \in G.$$ 

**Theorem (McKee, Todorov & T, ’18)**

Let $(A, G, \alpha)$ be a C*-dynamical system and $F : G \to CB(A)$ be a pointwise measurable map. TFAE:

1. $F$ is a Herz-Schur $(A, G, \alpha)$-multiplier;
2. $\mathcal{N}(F)$ is a Schur $A$-multiplier.

Idea: Similar to Jolissaint’s idea for the case $A = \mathbb{C}$ and uses Haagerup-Paulsen-Wittstock theorem and the characterization of Schur $A$-multipliers.
**Transference Result**

For \( F : G \to CB(A) \), let \( N(F) : G \times G \to CB(A) \) be given by

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**Embedding into Schur $A$-multipliers**

Let $G$ be second countable or discrete and let $\mathcal{T}(\varphi) : G \times G \to CB(A)$ be the function given by

$$\mathcal{T}(\varphi)(s, t)(a) = \alpha_t(\varphi(s, t)(\alpha_{t^{-1}}(a))), \quad a \in A.$$ 

**Definition**

A Schur $A$-multiplier $\varphi : G \times G \to CB(A)$ will be called **invariant** if for each $r \in G$

$$\mathcal{T}(\varphi)(s, t) = \mathcal{T}(\varphi)(sr, tr), \quad a.e.$$ 

Let $\mathcal{G}_{inv}(G, G; A)$ be the set of all invariant Schur $A$-multipliers; $\varphi \in \mathcal{G}_{inv}(G, G; A)$ if $S_\varphi$ commutes with $\tilde{\alpha}_t = \text{Ad}_\rho_t \otimes \alpha_t$, $t \in G$, where $\rho$ is the right regular representation.

**Theorem (McKee, Todorov & T ’18)**

The map $\mathcal{N}$ is a complete isometry from $\mathcal{G}(A, G, \alpha)$ onto $\mathcal{G}_{inv}(G, G; A)$. 
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A Schur $A$-multiplier $\varphi: G \times G \to CB(A)$ will be called **invariant** if for each $r \in G$

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Let $\mathcal{G}_{inv}(G, G; A)$ be the set of all invariant Schur $A$-multipliers; $\varphi \in \mathcal{G}_{inv}(G, G; A)$ if $S_\varphi$ commutes with $\tilde{\alpha}_t = Ad_\rho_t \otimes \alpha_t$, $t \in G$, where $\rho$ is the right regular representation.

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Classes of Herz-Schur multipliers: multiplication multipliers

Proposition

Let \( u : G \to \mathbb{C} \) be a bounded continuous function, and let \( F_u : G \to CB(A) \) be given by \( F_u(t)(a) = u(t)a, \ a \in A, \ t \in G \). TFAE:

1. \( F_u \) is a Herz-Schur \( (A, G, \alpha) \)-multiplier;
2. \( u \in M^{cb}A(G) \).
CLASSES OF HERZ-SCHUR MULTIPLIERS: CENTRAL MULTIPLIERS

A Schur $A$-multiplier $\varphi$ is called central if there exists a family $(a_{x,y})_{(x,y)\in X \times Y} \subset Z(A)$ (the center of $A$) such that

$$\varphi(x, y)(a) = a_{x,y}a, \ a \in A.$$ 

A Herz-Schur multiplier $F$ of $(A, G, \alpha)$ is central if

$$F(s)(a) = a_s a, \ a \in A$$

for some family $(a_s)_{s \in G} \in Z(A)$.

- If $F$ is a central Herz-Schur multiplier of $(A, G, \alpha)$, $S_F$ is $A$-bimodule. Those maps were considered by Dong-Ruan when studied Hilbert $A$-module Haagerup property of crossed product $A \rtimes_{\alpha, r} G$.
- Schur multipliers give rise to central Herz-Schur multipliers of $(\ell^\infty(G), G, \alpha)$:

$$\varphi \in \mathcal{S}(G, G) \rightsquigarrow a_s(p) = \varphi(s^{-1}p^{-1}, p^{-1}) \in \mathcal{S}_{cent}(\ell^\infty(G), G, \alpha)$$
**Classes of Herz-Schur multipliers: central multipliers**

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Theorem (McKee, Pourshahami, Todorov & T)

TFAE

- $\varphi : X \times Y \to Z(A)$ is a central Schur $A$-multiplier.
- There exist measurable $\alpha_i : X \to Z(A)''$ and $\beta_i : Y \to Z(A)''$ such that

$$\text{esssup}_{x \in X} \| \sum_i \alpha_i^*(x) \alpha_i(x) \| \text{esssup}_{y \in Y} \| \sum_i \beta_i^*(y) \beta_i(y) \| < \infty$$

and $\varphi(x, y) = \sum_{i \in I} \alpha_i(x) \beta_i(y)$ a.e. on $X \times Y$.

Let $E, F$ be operator spaces and $M$ an injective von Neumann algebra. If $\omega : F \otimes_h E \to M$ is a completely bounded map then there exist two families

$$\alpha = (\alpha_i)_{i \in I} \in C^\omega_I(CB(E, M)), \beta = (\beta_i)_{i \in I} \in C^\omega_I(CB(F, M))$$

such that $\|\alpha\|_{cb} \|b\|_{cb} = \|\omega\|_{cb}$ and $\omega(y \otimes x) = \sum_i \beta_i(y) \alpha_i(x), x \in E, y \in F$. 

Smith-Sinclair ’98, Le Merdy, Todorov & T ’19
Theorem (McKee, Pourshahami, Todorov & T)

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and $\varphi(x, y) = \sum_{i \in I} \alpha_i(x) \beta_i(y)$ a.e. on $X \times Y$.

$\varphi \in \mathcal{G}_{\text{cent}}(X, Y, A) \leadsto \Phi_\varphi : L^1(X) \otimes_h L^1(Y) \to Z(A)^{''}$ a cb map. Then the Smith-Sinclair factorization theorem gives the factorization:

Smith-Sinclair '98, Le Merdy, Todorov & T ’19

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Corollary (McKee, Pourshahami, Todorov & T)

Let \((C(X), G, \alpha)\) be a dynamical system. \(F : G \rightarrow C(X), \ F(r)(a) = a_r a, \ a_r \in C(X), \ r \in G. \) Then \(F\) is a Herz-Schur \((C(X), G, \alpha)\) multiplier \((\Leftrightarrow F \in M_{cb}(\mathcal{G}), \text{where } \mathcal{G} \text{ is the corresponding groupoid})\) iff there exists a Hilbert space \(K\) and weakly measurable functions \(\alpha, \beta : G \times X \rightarrow K\) such that

\[
a_{ts^{-1}}(xt^{-1}) = \langle \alpha(s, x), \beta(t, x) \rangle_K \text{ a.e. on } G \times G \times X.
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Question:

What are contractive idempotent central Schur multipliers/ central Herz-Schur multipliers?
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What are contractive idempotent central Schur multipliers/ central Herz-Schur multipliers?
Contractive Schur and Herz-Schur idempotents

Contractive Schur multipliers have been characterized by Katavolos, Paulsen ’05. A set $E \subset X \times Y$ says to have the 3-of-4 property if whenever 3 of 4 ordered pairs $(x_i, y_j), i, j = 1, 2$, belong to $E$ then the fourth pair belong to $E$.

Livshits’ observation: If $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $S_A : M_2 \to M_2$ is the map given as Schur product by $A$ then $\|S_A\| = 2/\sqrt{3}$. It gives

Theorem (Katavolos, Paulsen ’05)

- $\chi_E$, where $E \subset \mathbb{N} \times \mathbb{N}$, is a Schur multiplier of norm $< 2/\sqrt{3}$ iff $\|\chi_E\|_m = 1$ and $E$ has the 3-of-4 property and hence $E = \bigcup_{n=1}^{\infty} I_m \times J_m$ with $\{I_m\}$, $\{J_m\}$ countable collections of disjoint subsets of $\mathbb{N}$.

- Let $(X, \mu)$ be a standard measure space. Then $\chi_E(x, y)$ is a contractive idempotent Schur multiplier iff $E \simeq \bigcup_{n=1}^{\infty} A_n \times B_n$ (marginally equivalent), where $\{A_n\}_n$, $\{B_n\}_n$ are countable collections of disjoint Borel subsets of $X$. 
**Theorem** [McKee, Pourshahami, Todorov & T]

Let \( \varphi : X \times Y \times Z \to \mathbb{C} \) be measurable and continuous in \( z \)-variable. TFAE

- \( \varphi \) is a contractive idempotent Schur \( C(Z) \)-multiplier
- \( \varphi(x, y, z) = \sum_{i=1}^{\infty} \chi_{A_i^z}(x) \chi_{B_i^z}(y), (x, y) \text{ a.e., } z \in Z, \) where \( \{A_i^z\}, \{B_i^z\} \) are disjoint Borel sets.

In discrete case \( \varphi = \chi_W \) is a central contractive Schur \( C(Z) \)-multiplier iff \( W_z = \{(x, y) : (x, y, z) \in W\} \) has the 3-of-4 property for each \( z \).

Using the connection between Schur multipliers and Herz-Schur multipliers, Popa Stan ’09 proved

**Theorem (Popa Stan ’09)**

*Let \( G \) be a locally compact group and \( A \subset G \). Then \( \chi_A \) is a contractive idempotent Herz-Schur multiplier iff \( A \) is an open coset in \( G \).*
**Theorem [McKee, Pourshahami, Todorov & T]**

Let \( \varphi : X \times Y \times Z \to \mathbb{C} \) be measurable and continuous in \( z \)-variable. TFAE

- \( \varphi \) is a contractive idempotent Schur \( C(Z) \)-multiplier
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**Theorem (Popa Stan ’09)**

*Let \( G \) be a locally compact group and \( A \subset G \). Then \( \chi_A \) is a contractive idempotent Herz-Schur multiplier iff \( A \) is an open coset in \( G \).*
Let $G$ be a locally s.c. compact group acting on a locally compact space $X$, write $\alpha_t(x) = xt, x \in X, t \in G$. The set $\mathcal{G} = X \times G$ is a groupoid. The set of composable pairs is

$$\mathcal{G}^2 = \{[(x_1, t_1), (x_2, t_2)] : x_2 = x_1 t_1\}$$

the inverse $(x, t)^{-1}$ is defined by $(xt, t^{-1})$.

**Corollary**[McKee, Pourshahami, Todorov & T]

Let $V \subset X \times G$ be a clopen subset. Then $\chi_V$ is a contractive Herz-Schur multiplier of $(C(X), G, \alpha)$ iff $(x, t), (y, s)$ and $(z, p) \in V$ and $(z, p)(y, s)^{-1}(x, t)$ is well defined then the product belongs to $V$. 
**Convolution Multipliers**

Let $M(G)$ the measure algebra with convolution. There exists a complete isometry (Ghahramani ’78, Neufang ’00)

\[ \Theta : M(G) \rightarrow CB_{VN(G)}^{\sigma,L_\infty(G)}(B(L^2(G))), \]

\[ \Theta(\mu)(T) = \int_R \rho_s T \rho_s^* d\mu(s), T \in B(L^2(G)). \]

Let $\beta \in \text{Aut}(C_0(G))$ be given by $\beta_t(f)(x) = f(t^{-1}x)$ and consider $C_0(G) \rtimes_{r,\beta} G$. For $\Lambda = (\mu_t)_{t \in G} \subset M(G)$ let

\[ F(t)(f) = \Theta(\mu_t)(f) = f \ast \mu_t. \]

Set $S_{\text{conv}}(L^\infty(G), G, \beta)$ to be the set of all such Herz-Schur multipliers so that $S_F$ extends weak$^*$ to $L^\infty(G) \rtimes_{r,\beta} G$.

**Question**

What are the convolution multipliers?
CONVOLUTION MULTIPLIERS

Let $M(G)$ the measure algebra with convolution. There exists a complete isometry (Ghahramani '78, Neufang '00)

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$$F(t)(f) = \Theta(\mu_t)(f) = f \ast \mu_t.$$ 

Set $\mathcal{G}_{\text{conv}}(L^\infty(G), G, \beta)$ to be the set of all such Herz-Schur multipliers so that $S_F$ extends weak$^*$ to $L^\infty(G) \rtimes_{r,\beta} G$.

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**CONVOLUTION MULTIPLIERS**

Let $M(G)$ the measure algebra with convolution. There exists a complete isometry (Ghahramani ’78, Neufang ’00)

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Set $\mathcal{G}_{conv}(L^\infty(G), G, \beta)$ to be the set of all such Herz-Schur multipliers so that $S_F$ extends weak* to $L^\infty(G) \rtimes_{r,\beta} G$.

**Question**

What are the convolution multipliers?
Any \( F(t) = u(t)\Theta(\mu) = \Theta(u(t)\mu), t \in G, u \in M_{cb}(A(G)), \mu \in M(G), \) is such.

**Theorem [McKee, Pourshahami, Todorov & T]**

The map \( S : G_{conv}(L^\infty(G), G, \beta) \to M^r_{cb}(L^1(G) \hat{\otimes} A(G)) \)

\[
F(t) = \Theta(\mu_t) \mapsto R(f \otimes v)(s, t) := f \ast \mu_t(s) \otimes v(t), f \in L^1(G), v \in A(G)
\]

is a completely contractive isomorphism.

Here \( M^r_{cb}(L^1(G) \hat{\otimes} A(G)) \) is the set of right completely bounded multipliers \( R \) of \( L^1(G) \hat{\otimes} A(G) \), i.e. \( R(ab) = aR(b), a, b \in L^1(G) \hat{\otimes} A(G) \) and \( R \) is c.b.

- If \( R \in M^r_{cb}(L^1(G) \hat{\otimes} A(G)) \) then for the dual \( R^* \in CB^\sigma(L^\infty(G) \hat{\otimes} VN(G)) \) there exists \( (\mu_t)_{t \in G} \) such that
  \[
  R^*(f \otimes \lambda_t) = \Theta(\mu_t)(f) \otimes \lambda_t
  \]

- \( S_F \in CB^\sigma(L^\infty(G) \rtimes_r \beta G) \) can be lifted to a normal c.b. map on \( L^\infty(G) \hat{\otimes} VN(G) \) which is a right \( L^1(G) \hat{\otimes} A(G) \)-module map.

- Use Junge-Neufang-Ruan result ’09 to get \( \Phi_R \in CB^\sigma(B(L^2(G) \otimes L^2(G))) \) associated to \( R \in M_{cb}(L^1(G) \hat{\otimes} A(G)) \) which is a \( VN(G) \hat{\otimes} L^\infty(G) \)-bimodule map. \( \Phi_R|_{L^\infty(G) \rtimes G} = S_F \) for convolution multiplier \( F \) related to \( R \).
Any $F(t) = u(t)\Theta(\mu) = \Theta(u(t)\mu), t \in G, u \in M_{cb}(A(G)), \mu \in M(G)$, is such.

**Theorem** [McKee, Pourshahami, Todorov & T]

The map $S : \mathcal{S}_{\text{conv}}(L^\infty(G), G, \beta) \to M^r_{cb}(L^1(G)\hat{\otimes}A(G))$

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Question

What are contractive idempotent convolution multipliers?

- If \( G \) is commutative, then \( M_{cb}(L^1(G) \hat{\otimes} A(G)) = B(\hat{G} \times G) \), the Fourier-Stiltjes algebra of \( \hat{G} \times G \). Any contractive idempotent of \( B(\hat{G} \times G) \) is \( \chi_C \) where \( C \) is an open coset (\( \hat{\mu}(s) = \chi_C(s, t) \)). What are open subgroups of \( \hat{G} \times G \)?

- If \( G \) is a general l.c.group, any contractive idempotent measure \( \mu \in M(G) = M_{cb}^r(L^1(G)) \) is given by \( \mu = \gamma m_H \), where \( m_H \) is the Haar measure of a compact subgroup \( H \) and \( \gamma \) is a character of \( H \) (Greenleaf ’65). Hence \( \gamma m_H \otimes \chi_C \in M_{cb}^r(L^1(G) \hat{\otimes} A(G)) \), \( C \) is an open coset of \( G \). The corresponding convolution multiplier is given by \( \mu_t = \chi_C(t)\gamma m_H \).

Are there other contractive idempotents?
Question

What are contractive idempotent convolution multipliers?

- If $G$ is commutative, then $M_{cb}(L^1(G) \hat{\otimes} A(G)) = B(\hat{G} \times G)$, the Fourier-Stiltjes algebra of $\hat{G} \times G$. Any contractive idempotent of $B(\hat{G} \times G)$ is $\chi_C$ where $C$ is an open coset ($\hat{\mu}_t(s) = \chi_C(s,t)$). What are open subgroups of $\hat{G} \times G$?

- If $G$ is a general l.c.group, any contractive idempotent measure $\mu \in M(G) = M^r_{cb}(L^1(G))$ is given by $\mu = \gamma m_H$, where $m_H$ is the Haar measure of a compact subgroup $H$ and $\gamma$ is a character of $H$ (Greenleaf ’65). Hence $\gamma m_H \otimes \chi_C \in M^r_{cb}(L^1(G) \hat{\otimes} A(G))$, $C$ is an open coset of $G$. The corresponding convolution multiplier is given by $\mu_t = \chi_C(t) \gamma m_H$. Are there other contractive idempotents?
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- If $G$ is commutative, then $M_{cb}(L^1(G) \hat{\otimes} A(G)) = B(\hat{G} \times G)$, the Fourier-Stiltjes algebra of $\hat{G} \times G$. Any contractive idempotent of $B(\hat{G} \times G)$ is $\chi_C$ where $C$ is an open coset ($\hat{\mu}_t(s) = \chi_C(s, t)$). **What are open subgroups of $\hat{G} \times G$?**

- If $G$ is a general l.c.group, any contractive idempotent measure $\mu \in M(G) = M'_{cb}(L^1(G))$ is given by $\mu = \gamma m_H$, where $m_H$ is the Haar measure of a compact subgroup $H$ and $\gamma$ is a character of $H$ (Greenleaf '65). Hence $\gamma m_H \otimes \chi_C \in M'_{cb}(L^1(G) \hat{\otimes} A(G))$, $C$ is an open coset of $G$. The corresponding convolution multiplier is given by $\mu_t = \chi_C(t) \gamma m_H$. **Are there other contractive idempotents?**
**APPROXIMATIONS**

Let $G$ be a discrete group.

- It has been known that the existence of Herz-Schur multipliers of a particular type encodes various approximation properties of $C^*_r(G)$ (nuclearity, CBAP, exactness,...): if $(\phi_i)_i$ is a net of Herz-Schur multipliers with certain properties then $(S_{\phi_i})_i$ implements an approximation property of $C^*_r(G)$; conversely any family of approximating maps on $C^*_r(G)$ can be 'averaged' into Herz-Schur multipliers.

- One can expect that Herz-Schur multipliers of $(A, G, \alpha)$ will give approximations in $A \rtimes_{r,\alpha} G$.

- We have studied so far
  - Nuclearity (McKee, Skalski, Todorov & T, ’18)
  - The Haagerup property (McKee, Skalski, Todorov & T, ’18)
  - CBAP (McKee, ’18)
  - Exactness (McKee & T, ’19)
  - Compactness/complete compactness of Herz-Schur multipliers of $(A, G, \alpha)$ (He, Todorov & T ’19)
**Approximations**

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**Complete compactness of Herz-Schur multipliers**

Let $\mathcal{X}$ be an operator space. A completely bounded map $\Psi : \mathcal{X} \to \mathcal{X}$ ($\Psi \in CC(\mathcal{X})$) is called *completely compact* (due to Wittstock, Saar ’82) if for every $\varepsilon > 0$ there exists a finite dimensional subspace $\mathcal{Y} \subset \mathcal{X}$ such that

$$\text{dist}(\Psi^{(m)}(x), M_m(\mathcal{Y})) < \varepsilon \quad \forall x \in M_m(\mathcal{X}), \quad \|x\| \leq 1 \quad \forall m \in \mathbb{N}.$$ 

- Finite rank maps are completely compact.
- $CC(\mathcal{X})$ is a closed ideal in $CB(\mathcal{X})$.
- If $\mathcal{X}$ has the completely bounded approximation property (CBAP), i.e. $\text{id} : \mathcal{X} \to \mathcal{X}$ can be approximated in the point norm topology by finite rank maps $\Phi_\alpha \in CB(\mathcal{X})$, $\sup_{\alpha} \|\Phi_\alpha\| < \infty$ then $F(\mathcal{X}) = CC(\mathcal{X})$.

**Problem**

Characterize completely compact Herz-Schur multipliers/Herz-Schur multipliers of dynamical systems.
Complete compactness of Herz-Schur multipliers

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**COMPLETE COMPACTNESS OF HERZ-SCHUR MULTIPLIERS**

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Characterize completely compact Herz-Schur multipliers/Herz-Schur multipliers of dynamical systems.
Theorem [He, Todorov & T, ’19]

Let $F$ be a Herz-Schur multiplier of $(A, G, \alpha)$. Assume $A \rtimes_{r, \alpha} G$ has the CBAP. Then TFAE

1. $S_F$ is completely compact

2. There is $(F_i)_i \in \mathcal{S}(A, G, \alpha)$ such that each $F_i$ is finitely supported, $F_i(s) \in F(A), s \in G, \sup_i \|F_i\|_m < \infty$ and $\|F_i - F\|_m \to 0$

3. There is a sequence $(\varphi_i)_i$ of band finite Schur $A$-multipliers and f.d. spaces $\mathcal{Y}_i \subset A$ s.t. $\operatorname{Ran}(\alpha_t \circ \varphi_i(s, t) \circ \alpha_{t^{-1}}) \subset \mathcal{Y}_i, s, t \in G$, $\sup_i \|\varphi_i\|_\mathcal{S} < \infty$ and $\|N(F) - \varphi_i\|_m \to 0$.

4. If $A = \mathbb{C}$ then $F \in \overline{A(G)}_{\|\cdot\|_{M_{cb}(A(G))}}$.

(1) $\Rightarrow$ (2): $A \rtimes_{r, \alpha} G$ has the CBAP $\leadsto \{\Phi_\alpha\}$ finite rank, $\sup_\alpha \|\Phi_\alpha\| < \infty, \Phi_\alpha \to \text{id}$ in the point norm topology and $\operatorname{Ran}(\Phi_\alpha)$ in $\text{span}\{\pi(a)\lambda_s : a \in A, s \in G\}$.

Let $\pi(F_\alpha(s)(a)) = \mathcal{E}(\Phi_\alpha(\pi(a)\lambda_s)\lambda_s^*)$ where $\mathcal{E}$ is a faithful conditional expectation from $A \rtimes_{r, \alpha} G$ to $A$. A subsequence of $\{F_\alpha \circ F\}$ will do the job.

(1) $\iff$ (4) is true if $G$ has the AP.
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1. $S_F$ is completely compact
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(1) $\Leftrightarrow$ (4) is true if $G$ has the AP.
Bozejko ’82 constructed $\varphi_n$ with finite supports $E_n \subset F_\infty$, the free group, such that $\|\varphi_n\|_{MA(F_\infty)} = 1$ and $\|\varphi_n\|_{M_{cb}A(F_\infty)} \geq C\sqrt{n}$. This gives

**Proposition**

If $G$ is a discrete group containing $F_\infty$ then there exists a compact Herz-Schur multiplier which is not completely compact.

**Problem**

Find a completely bounded compact Herz-Schur multiplier which is not completely compact.

Saar ’82 constructed a completely bounded map on $\mathcal{K}(H)$ which is compact but not completely compact.

**Example**

If $T : A \rightarrow A$ is a completely compact such that $\alpha_t \circ T = T \circ \alpha_t$, $t \in G$, and $u \in \overline{A(G)}\|\cdot\|_{M_{cb}(A(G))}$ then $F(t) = u(t)T(a)$, $a \in A$, is a completely compact multiplier of $A \rtimes_{r,\alpha} G$.

In particular, if $T \in \overline{F(A)}$ and $G$ acts amenably on $A$ then $S_F$ can be approximated in cb norm by finite rank operators $S_{F_i}$, $(F_i)_i \subset \mathcal{S}(A, G, \alpha)$. 
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THANK YOU!