Wittstock moduli of elementary operators and their application to generalized notions of amenability

Volker Runde

University of Alberta

Winnipeg, July, 2019
Wittstock moduli, I

Notation

Notation

$A, B$: $C^*$-algebras.

$\text{CB}(A, B)$: completely bounded linear maps $A \rightarrow B$.

$\text{CP}(A, B)$: completely positive linear maps $A \rightarrow B$.

Remarks

1. $\text{CP}(A, B) \subset \text{CB}(A, B)$.

2. For $A$ unital and $T \in \text{CP}(A, B)$:
   $$\|T\|_{cb} = \|T\| = \|T(e_A)\|.$$
Wittstock moduli, I

Notation

\(A, B:\ C^*\)-algebras.
### Notation

$\mathcal{A}, \mathcal{B}$: $C^*$-algebras.

$\mathcal{CB}(\mathcal{A}, \mathcal{B})$: completely bounded linear maps $\mathcal{A} \rightarrow \mathcal{B}$; $\mathcal{CP}(\mathcal{A}, \mathcal{B})$: completely positive linear maps $\mathcal{A} \rightarrow \mathcal{B}$.

Remarks:
1. $\mathcal{CP}(\mathcal{A}, \mathcal{B}) \subset \mathcal{CB}(\mathcal{A}, \mathcal{B})$.
2. For $\mathcal{A}$ unital and $T \in \mathcal{CP}(\mathcal{A}, \mathcal{B})$:
   \[
   \|T\|_{\text{cb}} = \|T\|_{\text{ps}} = \|T(e_\mathcal{A})\|
   \]
Wittstock moduli of elementary operators and their application to generalized notions of amenability

Volker Runde

Notation

\(\mathcal{A}, \mathcal{B}: \text{C}^*\)-algebras.

\(\mathcal{CB}(\mathcal{A}, \mathcal{B})\): completely bounded linear maps \(\mathcal{A} \to \mathcal{B}\).
Wittstock moduli, I

Notation

\( \mathcal{A}, \mathcal{B} \): \( C^\ast \)-algebras.

\( \text{CB}(\mathcal{A}, \mathcal{B}) \): completely bounded linear maps \( \mathcal{A} \to \mathcal{B} \);

\( \text{CP}(\mathcal{A}, \mathcal{B}) \):
Wittstock moduli, I

**Notation**

- $\mathcal{A}, \mathcal{B}$: C*-algebras.
- $\mathcal{CB}(\mathcal{A}, \mathcal{B})$: completely bounded linear maps $\mathcal{A} \rightarrow \mathcal{B}$;
- $\mathcal{CP}(\mathcal{A}, \mathcal{B})$: completely positive linear maps $\mathcal{A} \rightarrow \mathcal{B}$.
Wittstock moduli, I

Notation

\(A, \ B: \ \text{C}^*-\text{algebras.}\)

\(\text{CB}(A, B): \ \text{completely bounded linear maps } A \to B;\)

\(\text{CP}(A, B): \ \text{completely positive linear maps } A \to B.\)

Remarks
Notation

\( \mathcal{A}, \mathcal{B} \): \( \mathcal{C}^* \)-algebras.

\( \text{CB}(\mathcal{A}, \mathcal{B}) \): completely bounded linear maps \( \mathcal{A} \to \mathcal{B} \);

\( \text{CP}(\mathcal{A}, \mathcal{B}) \): completely positive linear maps \( \mathcal{A} \to \mathcal{B} \).

Remarks

1. \( \text{CP}(\mathcal{A}, \mathcal{B}) \subset \text{CB}(\mathcal{A}, \mathcal{B}) \).
Wittstock moduli, I

Notation

$A, B$: $C^*$-algebras.

$CB(A, B)$: completely bounded linear maps $A \rightarrow B$;

$CP(A, B)$: completely positive linear maps $A \rightarrow B$.

Remarks

1. $CP(A, B) \subset CB(A, B)$.
2. For $A$ unital
Wittstock moduli, I

**Notation**

\( \mathcal{A}, \mathcal{B} \): \( C^* \)-algebras.

\( \mathcal{CB}(\mathcal{A}, \mathcal{B}) \): completely bounded linear maps \( \mathcal{A} \to \mathcal{B} \);

\( \mathcal{CP}(\mathcal{A}, \mathcal{B}) \): completely positive linear maps \( \mathcal{A} \to \mathcal{B} \).

**Remarks**

1. \( \mathcal{CP}(\mathcal{A}, \mathcal{B}) \subset \mathcal{CB}(\mathcal{A}, \mathcal{B}) \).

2. For \( \mathcal{A} \) unital and \( T \in \mathcal{CP}(\mathcal{A}, \mathcal{B}) \):
Notation

\( \mathcal{A}, \mathcal{B} \): C*-algebras.

\( \mathcal{CB}(\mathcal{A}, \mathcal{B}) \): completely bounded linear maps \( \mathcal{A} \to \mathcal{B} \);
\( \mathcal{CP}(\mathcal{A}, \mathcal{B}) \): completely positive linear maps \( \mathcal{A} \to \mathcal{B} \).

Remarks

1. \( \mathcal{CP}(\mathcal{A}, \mathcal{B}) \subseteq \mathcal{CB}(\mathcal{A}, \mathcal{B}) \).
2. For \( \mathcal{A} \) unital and \( T \in \mathcal{CP}(\mathcal{A}, \mathcal{B}) \):

\[ \| T \|_{cb} = \| T \| = \| T(e_{\mathcal{A}}) \|. \]
Wittstock moduli of elementary operators and their application to generalized notions of amenability

Volker Runde

Wittstock moduli of elementary operators

Generalized notions of amenability

Paulsen’s off-diagonal technique

Conclusion

Advertisements
\[ A, B: C^\ast\text{-algebras,} \]
\( \mathbf{A}, \mathbf{B} : \text{C}^*\text{-algebras, } T \in \text{CB}(\mathbf{A}, \mathbf{B}). \)
Wittstock moduli, II

\[ A, B: C^*-\text{algebras}, \ T \in CB(A, B). \text{ Define } T^* \in CB(A, B): \]

\[ a, b : C^*-\text{algebras}, \ T \in CB(A, B). \text{ Define } T^* \in CB(A, B): \]
\( \mathcal{A}, \mathcal{B} \): \( C^* \)-algebras, \( T \in CB(\mathcal{A}, \mathcal{B}) \). Define \( T^* \in CB(\mathcal{A}, \mathcal{B}) \):

\[
T^*(a) := T(a^*)^* \quad (a \in \mathcal{A}).
\]
\( \mathcal{A}, \mathcal{B}: \text{C}^*\text{-algebras, } T \in \text{CB}(\mathcal{A}, \mathcal{B}). \text{ Define } T^* \in \text{CB}(\mathcal{A}, \mathcal{B}): \\
T^*(a) := T(a^*)^* \quad (a \in \mathcal{A}). \)

Set

\[
\text{Re } T := \frac{1}{2}(T + T^*)
\]
\( \mathcal{A}, \mathcal{B} \): \( C^* \)-algebras, \( T \in CB(\mathcal{A}, \mathcal{B}) \). Define \( T^* \in CB(\mathcal{A}, \mathcal{B}) \):

\[
T^*(a) := T(a^*)^* \quad (a \in \mathcal{A}).
\]

Set

\[
\text{Re } T := \frac{1}{2}(T + T^*)
\]

and

\[
\text{Im } T := \frac{1}{2i}(T - T^*).
\]
\( \mathcal{A}, \mathcal{B} \): \( C^* \)-algebras, \( T \in \mathcal{CB}(\mathcal{A}, \mathcal{B}) \). Define \( T^* \in \mathcal{CB}(\mathcal{A}, \mathcal{B}) \):

\[
T^*(a) := T(a^*)^* \quad (a \in \mathcal{A}).
\]

Set

\[
\text{Re } T := \frac{1}{2}(T + T^*)
\]

and

\[
\text{Im } T := \frac{1}{2i}(T - T^*).
\]

Then

\[
T = \text{Re } T + i \text{ Im } T.
\]
Wittstock moduli, III

Wittstock moduli of elementary operators and their application to generalized notions of amenability

Volker Runde

Wittstock's Decomposition Theorem (Wittstock, 1981 & Paulsen, 1982 & Haagerup, ≈ same time)

Every $T \in CB(A, B(H))$ has a Wittstock modulus $|T|$. 

Remarks

1 $|T|$ need not be unique.

2 If $A$ is unital, we can find $|T|$ with $|T|(e_A) = \|T\|_{cb id H}$. 

Definitions

Let $A$, $B$ be $C^*$-algebras, and let $T \in CB(A, B)$. We call $|T| \in CP(A, B)$ a Wittstock modulus for $T$ if:

1 $|T| \pm \text{Re} T$, $|T| \pm \text{Im} T \in CP(A, B)$.

Advertisements
Definition

Let $A$, $B$ be $C^*$-algebras, and let $T \in \mathcal{CB}(A, B)$. We call $|T| \in \mathcal{CP}(A, B)$ a Wittstock modulus for $T$ if:

1. $|T| \pm \text{Re} T, |T| \pm \text{Im} T \in \mathcal{CP}(A, B)$.

Wittstock's Decomposition Theorem (Wittstock, 1981 & Paulsen, 1982 & Haagerup, same time)

Every $T \in \mathcal{CB}(A, B(H))$ has a Wittstock modulus $|T|$.

Remarks

1. $|T|$ need not be unique.

2. If $A$ is unital, we can find $|T|$ with $|T|(e_A) = \|T\|_{cb}$. 

Advertisements
Definition

Let $A, B$ be $C^*$-algebras,
Wittstock moduli, III

Definition

Let $\mathcal{A}, \mathcal{B}$ be $C^*$-algebras, and let $T \in CB(\mathcal{A}, \mathcal{B})$. 

1. $|T| \pm \text{Re} T, |T| \pm \text{Im} T \in CP(\mathcal{A}, \mathcal{B})$.

Wittstock's Decomposition Theorem (Wittstock, 1981 & Paulsen, 1982 & Haagerup, same time)

Every $T \in CB(\mathcal{A}, \mathcal{B}(H))$ has a Wittstock modulus $|T|$.

Remarks

1. $|T|$ need not be unique.
2. If $\mathcal{A}$ is unital, we can find $|T|$ with $|T|(e_{\mathcal{A}}) = \|T\|_{cb \text{id} H}$. 

Advertisements
Wittstock moduli, III

Definition

Let $\mathcal{A}, \mathcal{B}$ be $C^*$-algebras, and let $T \in CB(\mathcal{A}, \mathcal{B})$. We call $|T| \in CP(\mathcal{A}, \mathcal{B})$
**Definition**

Let $\mathcal{A}, \mathcal{B}$ be $C^*$-algebras, and let $T \in CB(\mathcal{A}, \mathcal{B})$. We call $|T| \in CP(\mathcal{A}, \mathcal{B})$ a Wittstock modulus for $T$. 

Wittstock's Decomposition Theorem (Wittstock, 1981 & Paulsen, 1982 & Haagerup, same time)

Every $T \in CB(\mathcal{A}, \mathcal{B}(H))$ has a Wittstock modulus $|T|$.

Remarks

1. $|T|$ need not be unique.
2. If $\mathcal{A}$ is unital, we can find $|T|$ with $|T|(e_A) = \|T\|_{cb}$. 

Advertisements
Definition

Let $\mathcal{A}, \mathcal{B}$ be $C^*$-algebras, and let $T \in \mathcal{CB}(\mathcal{A}, \mathcal{B})$. We call $|T| \in \mathcal{CP}(\mathcal{A}, \mathcal{B})$ a **Wittstock modulus** for $T$ if:
Wittstock moduli, III

Definition

Let $\mathcal{A}, \mathcal{B}$ be $C^*$-algebras, and let $T \in CB(\mathcal{A}, \mathcal{B})$. We call $|T| \in CP(\mathcal{A}, \mathcal{B})$ a **Wittstock modulus** for $T$ if:

1. $|T| \pm \text{Re } T, |T| \pm \text{Im } T \in CP(\mathcal{A}, \mathcal{B})$. 

Wittstock's Decomposition Theorem (Wittstock, 1981 & Paulsen, 1982 & Haagerup, ≈ same time)

Every $T \in CB(\mathcal{A}, \mathcal{B}(H))$ has a Wittstock modulus $|T|$.

Remarks

1. $|T|$ need not be unique.
2. If $\mathcal{A}$ is unital, we can find $|T|$ with $|T|(e_\mathcal{A}) = \|T\|_{cb}$. 

Advertisements
Wittstock moduli, III

Definition

Let \( \mathcal{A}, \mathcal{B} \) be \( C^* \)-algebras, and let \( T \in CB(\mathcal{A}, \mathcal{B}) \). We call \( |T| \in CP(\mathcal{A}, \mathcal{B}) \) a Wittstock modulus for \( T \) if:

1. \( |T| \pm \text{Re } T, |T| \pm \text{Im } T \in CP(\mathcal{A}, \mathcal{B}). \)

Wittstock’s Decomposition Theorem (Wittstock, 1981 & Paulsen, 1982 & Haagerup, \( \approx \) same time)
Wittstock moduli, III

**Definition**

Let $\mathcal{A}, \mathcal{B}$ be $C^*$-algebras, and let $T \in CB(\mathcal{A}, \mathcal{B})$. We call $|T| \in CP(\mathcal{A}, \mathcal{B})$ a *Wittstock modulus* for $T$ if:

1. $|T| \pm \text{Re } T, |T| \pm \text{Im } T \in CP(\mathcal{A}, \mathcal{B})$.

**Wittstock’s Decomposition Theorem (Wittstock, 1981 & Paulsen, 1982 & Haagerup, \approx same time)**

*Every $T \in CB(\mathcal{A}, B(\mathcal{H}))$*
Definition

Let $\mathcal{A}, \mathcal{B}$ be $C^*$-algebras, and let $T \in \text{CB}(\mathcal{A}, \mathcal{B})$. We call $|T| \in \mathcal{CP}(\mathcal{A}, \mathcal{B})$ a Wittstock modulus for $T$ if:

1. $|T| \pm \text{Re } T, |T| \pm \text{Im } T \in \mathcal{CP}(\mathcal{A}, \mathcal{B})$.

Wittstock’s Decomposition Theorem (Wittstock, 1981 & Paulsen, 1982 & Haagerup, $\approx$ same time)

Every $T \in \text{CB}(\mathcal{A}, \mathcal{B}(\mathcal{H}))$ has a Wittstock modulus $|T|$. 

Advertisements
Definition

Let $\mathcal{A}, \mathcal{B}$ be $C^*$-algebras, and let $T \in \mathcal{CB}(\mathcal{A}, \mathcal{B})$. We call $|T| \in \mathcal{CP}(\mathcal{A}, \mathcal{B})$ a Wittstock modulus for $T$ if:

1. $|T| \pm \text{Re } T, |T| \pm \text{Im } T \in \mathcal{CP}(\mathcal{A}, \mathcal{B})$.

Wittstock's Decomposition Theorem (Wittstock, 1981 & Paulsen, 1982 & Haagerup, $\approx$ same time)

Every $T \in \mathcal{CB}(\mathcal{A}, \mathcal{B}(\mathcal{H}))$ has a Wittstock modulus $|T|$.

Remarks
Definition

Let $\mathcal{A}, \mathcal{B}$ be $C^*$-algebras, and let $T \in CB(\mathcal{A}, \mathcal{B})$. We call $|T| \in CP(\mathcal{A}, \mathcal{B})$ a Wittstock modulus for $T$ if:

1. $|T| \pm \text{Re} T, |T| \pm \text{Im} T \in CP(\mathcal{A}, \mathcal{B})$.

Wittstock’s Decomposition Theorem (Wittstock, 1981 & Paulsen, 1982 & Haagerup, $\approx$ same time)

Every $T \in CB(\mathcal{A}, B(\mathcal{H}))$ has a Wittstock modulus $|T|$.

Remarks

1. $|T|$ need not be unique.
Wittstock moduli, III

**Definition**

Let $\mathcal{A}, \mathcal{B}$ be $C^*$-algebras, and let $T \in CB(\mathcal{A}, \mathcal{B})$. We call $|T| \in CP(\mathcal{A}, \mathcal{B})$ a **Wittstock modulus** for $T$ if:

1. $|T| \pm \text{Re}\ T, |T| \pm \text{Im}\ T \in CP(\mathcal{A}, \mathcal{B})$.

**Wittstock’s Decomposition Theorem** (Wittstock, 1981 & Paulsen, 1982 & Haagerup, $\approx$ same time)

Every $T \in CB(\mathcal{A}, \mathcal{B}(\mathcal{H}))$ has a Wittstock modulus $|T|$. 

**Remarks**

1. $|T|$ need not be unique.
2. If $\mathcal{A}$ is unital,
Wittstock moduli, III

Definition

Let \( \mathcal{A}, \mathcal{B} \) be \( C^* \)-algebras, and let \( T \in \text{CB}(\mathcal{A}, \mathcal{B}) \). We call \( |T| \in \text{CP}(\mathcal{A}, \mathcal{B}) \) a Wittstock modulus for \( T \) if:

1. \( |T| \pm \text{Re } T, |T| \pm \text{Im } T \in \text{CP}(\mathcal{A}, \mathcal{B}) \).

Wittstock’s Decomposition Theorem (Wittstock, 1981 & Paulsen, 1982 & Haagerup, \( \approx \) same time)

Every \( T \in \text{CB}(\mathcal{A}, \mathcal{B}(\mathcal{H})) \) has a Wittstock modulus \( |T| \).

Remarks

1. \( |T| \) need not be unique.
2. If \( \mathcal{A} \) is unital, we can find \( |T| \).
Wittstock moduli, III

Definition

Let \( \mathcal{A}, \mathcal{B} \) be \( C^* \)-algebras, and let \( T \in CB(\mathcal{A}, \mathcal{B}) \). We call \( |T| \in CP(\mathcal{A}, \mathcal{B}) \) a Wittstock modulus for \( T \) if:

1. \( |T| \pm Re \, T, |T| \pm Im \, T \in CP(\mathcal{A}, \mathcal{B}) \).

Wittstock’s Decomposition Theorem (Wittstock, 1981 & Paulsen, 1982 & Haagerup, ≈ same time)

Every \( T \in CB(\mathcal{A}, B(\mathcal{H})) \) has a Wittstock modulus \( |T| \).

Remarks

1. \( |T| \) need not be unique.
2. If \( \mathcal{A} \) is unital, we can find \( |T| \) with \( |T|(e_{\mathcal{A}}) = \| T \|_{cb} \, id_{\mathcal{H}} \).
Elementary operators, I

Definition

Let \( \text{id}_H \in A \subset B(\mathcal{H}) \) be a \( C^* \)-algebra, \( n \in \mathbb{N} \), and let \( a := (a_1, \ldots, a_n), b := (b_1, \ldots, b_n) \in A^n \).

Define \( M_{a,b} \in \mathcal{B}(B(\mathcal{H})) \) via

\[
M_{a,b}(x) := \sum_{j=1}^{n} a_j x b_j \quad (x \in B(\mathcal{H})).
\]

We call \( M_{a,b} \) an elementary operator on \( B(\mathcal{H}) \) with coefficients in \( A \).

Notation

\[
E^\ell A := \text{elementary operators on } B(\mathcal{H}) \text{ with coefficients in } A.
\]
Definition

Let $\text{id}_H \in A \subset B(H)$ be a $C^*$-algebra, let $n \in \mathbb{N}$, and let $a := (a_1, \ldots, a_n), b := (b_1, \ldots, b_n) \in A^n$. Define $M_{a, b} \in \text{CB}(B(H))$ via $M_{a, b}(x) := \sum_{j=1}^{n} a_j x b_j (x \in B(H))$. We call $M_{a, b}$ an elementary operator on $B(H)$ with coefficients in $A$. 

Notation

$E_\ell^A := \text{elementary operators on } B(H) \text{ with coefficients in } A$
Elementary operators, I

Definition

Let \( \text{id}_\mathcal{H} \in \mathcal{A} \subset B(\mathcal{H}) \) be a \( C^\ast \)-algebra,
Elementary operators, I

Definition

Let \( \text{id}_\mathcal{H} \in \mathcal{A} \subset \mathcal{B}(\mathcal{H}) \) be a \( C^* \)-algebra, let \( n \in \mathbb{N} \),
Elementary operators, I

Definition

Let \( \text{id}_\mathcal{H} \in \mathcal{A} \subset \mathcal{B}(\mathcal{H}) \) be a \( C^* \)-algebra, let \( n \in \mathbb{N} \), and let \( a := (a_1, \ldots, a_n), b := (b_1, \ldots, b_n) \in \mathcal{A}^n \).
Elementary operators, I

**Definition**

Let $\text{id}_{\mathcal{H}} \in \mathcal{A} \subset B(\mathcal{H})$ be a $C^*$-algebra, let $n \in \mathbb{N}$, and let $\alpha := (a_1, \ldots, a_n)$, $\beta := (b_1, \ldots, b_n) \in \mathcal{A}^n$. Define $M_{\alpha, \beta} \in CB(B(\mathcal{H}))$ via
Elementary operators, I

Definition

Let $\text{id}_\mathcal{H} \in \mathcal{A} \subset \mathcal{B}(\mathcal{H})$ be a $\mathcal{C}^*$-algebra, let $n \in \mathbb{N}$, and let $\mathbf{a} := (a_1, \ldots, a_n)$, $\mathbf{b} := (b_1, \ldots, b_n) \in \mathcal{A}^n$. Define $M_{\mathbf{a}, \mathbf{b}} \in \mathcal{CB}(\mathcal{B}(\mathcal{H}))$ via

$$M_{\mathbf{a}, \mathbf{b}}(x) := \sum_{j=1}^{n} a_j x b_j \quad (x \in \mathcal{B}(\mathcal{H})).$$
**Definition**

Let \( \text{id}_{\mathcal{H}} \in \mathcal{A} \subset B(\mathcal{H}) \) be a \( C^* \)-algebra, let \( n \in \mathbb{N} \), and let \( a := (a_1, \ldots, a_n), b := (b_1, \ldots, b_n) \in \mathcal{A}^n \). Define \( M_{a,b} \in CB(B(\mathcal{H})) \) via

\[
M_{a,b}(x) := \sum_{j=1}^{n} a_j x b_j \quad (x \in B(\mathcal{H})).
\]

We call \( M_{a,b} \)
Elementary operators, I

Definition

Let \( \text{id}_\mathcal{H} \in \mathcal{A} \subset B(\mathcal{H}) \) be a \( C^* \)-algebra, let \( n \in \mathbb{N} \), and let
\[
\alpha := (a_1, \ldots, a_n), \quad \beta := (b_1, \ldots, b_n) \in \mathcal{A}^n.
\]
Define
\[
M_{\alpha, \beta} \in CB(B(\mathcal{H})) \text{ via}
\]
\[
M_{\alpha, \beta}(x) := \sum_{j=1}^{n} a_j xb_j \quad (x \in B(\mathcal{H})).
\]

We call \( M_{\alpha, \beta} \) an elementary operator on \( B(\mathcal{H}) \).
Elementary operators, I

Definition

Let \( \text{id}_{\mathcal{H}} \in \mathcal{A} \subset B(\mathcal{H}) \) be a \( C^* \)-algebra, let \( n \in \mathbb{N} \), and let \( a := (a_1, \ldots, a_n), b := (b_1, \ldots, b_n) \in \mathcal{A}^n \). Define \( M_{a,b} \in CB(B(\mathcal{H})) \) via

\[
M_{a,b}(x) := \sum_{j=1}^{n} a_j x b_j \quad (x \in B(\mathcal{H})).
\]

We call \( M_{a,b} \) an elementary operator on \( B(\mathcal{H}) \) with coefficients in \( \mathcal{A} \).
Elementary operators, I

Definition

Let $\text{id}_{\mathcal{H}} \in \mathcal{A} \subset \mathcal{B}(\mathcal{H})$ be a $C^*$-algebra, let $n \in \mathbb{N}$, and let $\alpha := (a_1, \ldots, a_n), \beta := (b_1, \ldots, b_n) \in \mathcal{A}^n$. Define $M_{\alpha, \beta} \in \mathcal{C}\mathcal{B}(\mathcal{B}(\mathcal{H}))$ via

$$M_{\alpha, \beta}(x) := \sum_{j=1}^{n} a_j x b_j \quad (x \in \mathcal{B}(\mathcal{H})).$$

We call $M_{\alpha, \beta}$ an elementary operator on $\mathcal{B}(\mathcal{H})$ with coefficients in $\mathcal{A}$.

Notation
Elementary operators, I

**Definition**

Let $\text{id}_\mathcal{H} \in \mathcal{A} \subset \mathcal{B}(\mathcal{H})$ be a $C^*$-algebra, let $n \in \mathbb{N}$, and let $a := (a_1, \ldots, a_n), \ b := (b_1, \ldots, b_n) \in \mathcal{A}^n$. Define $M_{a,b} \in \mathcal{CB}(\mathcal{B}(\mathcal{H}))$ via

$$M_{a,b}(x) := \sum_{j=1}^{n} a_j x b_j \quad (x \in \mathcal{B}(\mathcal{H})).$$

We call $M_{a,b}$ an elementary operator on $\mathcal{B}(\mathcal{H})$ with coefficients in $\mathcal{A}$.

**Notation**

$$\mathcal{E} \mathcal{L}_\mathcal{A}(\mathcal{B}(\mathcal{H}))$$

$:= \text{elementary operators on } \mathcal{B}(\mathcal{H}) \text{ with coefficients in } \mathcal{A}$
Elementary operators, II

Wittstock moduli of elementary operators and their application to generalized notions of amenability
Volker Runde

Wittstock moduli of elementary operators
Generalized notions of amenability
Paulsen’s off-diagonal technique
Conclusion
Advertisements

1 $E_{\ell A}(B(H))$ is a subalgebra of $\mathcal{C}B(B(H))$.

2 $T \in E_{\ell A}(B(H)) \cap \mathcal{C}P(B(H))$ if and only if there are $n \in \mathbb{N}$, $c := (c_1, \ldots, c_n) \in A_n$ such that $T = M_{c, c^*}$ with $c^* := (c_1^*, \ldots, c_n^*)$.

3 $M_{a, b^*} = M_{b, a^*}$.

4 $E_{\ell A}(B(H)) \cong = A \otimes hA$. ($\otimes hA =$ Haagerup tensor product)

Question: Given $T \in E_{\ell A}(B(H))$, can a Wittstock modulus $|T| \in \mathcal{C}B(B(H))$ be found in $E_{\ell A}(B(H))$?

Answer: Yes!
Remarks

$E$\begin{align*}
\ell & \subset \mathcal{A}(\mathcal{B}(\mathcal{H})) \\
\text{iff} & \exists n \in \mathbb{N}, c := (c_1, \ldots, c_n) \in \mathcal{A}^n \\
& \text{such that} \\
T & = \mathcal{M}_{c, c^*} \quad \text{with} \\
c^* & := (c^*_1, \ldots, c^*_n).
\end{align*}

$E$\begin{align*}
\ell & \cong \mathcal{A} \otimes h \mathcal{A} \\
& \quad \text{Haagerup tensor product}
\end{align*}

Question: Given $T \in E$\begin{align*}
\ell & \subset \mathcal{B}(\mathcal{H}), \\
\text{can a Wittstock modulus} & |T| \in \mathcal{C} \mathcal{B}(\mathcal{B}(\mathcal{H})) \text{ be found}.
\end{align*}

Answer: Yes!
Remarks

1. \( \mathcal{E}_2(B(H)) \) is a subalgebra of \( CB(B(H)) \).
Remarks

1. $\mathcal{E}\ell_1(B(\mathcal{H}))$ is a subalgebra of $CB(B(\mathcal{H}))$.
2. $T \in \mathcal{E}\ell_1(B(\mathcal{H})) \cap CP(B(\mathcal{H}))$
Elementary operators, II

Remarks

1. \( \mathcal{E}_{\ell_2}(B(\mathcal{H})) \) is a subalgebra of \( CB(B(\mathcal{H})) \).
2. \( T \in \mathcal{E}_{\ell_2}(B(\mathcal{H})) \cap CP(B(\mathcal{H})) \) iff
Elementary operators, II

Remarks

1. $\mathcal{E}_{\ell_2}(B(\mathcal{H}))$ is a subalgebra of $CB(B(\mathcal{H}))$.
2. $T \in \mathcal{E}_{\ell_2}(B(\mathcal{H})) \cap CP(B(\mathcal{H}))$ iff there are $n \in \mathbb{N}$,
Remarks

1. \( \mathcal{E}_{\mathbb{A}}(\mathcal{B}(\mathcal{H})) \) is a subalgebra of \( \mathcal{CB}(\mathcal{B}(\mathcal{H})) \).

2. \( T \in \mathcal{E}_{\mathbb{A}}(\mathcal{B}(\mathcal{H})) \cap \mathcal{CP}(\mathcal{B}(\mathcal{H})) \) iff there are \( n \in \mathbb{N} \), \( \mathbf{c} := (c_1, \ldots, c_n) \in \mathbb{A}^n \) such that...
Elementary operators, II

Remarks

1. $\mathcal{E}_{\mathfrak{A}}(\mathcal{B}(\mathcal{H}))$ is a subalgebra of $\mathcal{CB}(\mathcal{B}(\mathcal{H}))$.

2. $T \in \mathcal{E}_{\mathfrak{A}}(\mathcal{B}(\mathcal{H})) \cap \mathcal{CP}(\mathcal{B}(\mathcal{H}))$ iff there are $n \in \mathbb{N}$, $c := (c_1, \ldots, c_n) \in \mathfrak{A}^n$ such that $T = M_{c,c^*}$.

Question

Given $T \in \mathcal{E}_{\mathfrak{A}}(\mathcal{B}(\mathcal{H}))$, can a Wittstock modulus $|T| \in \mathcal{CB}(\mathcal{B}(\mathcal{H}))$ be found in $\mathcal{E}_{\mathfrak{A}}(\mathcal{B}(\mathcal{H}))$?

Answer

Yes!
Elementary operators, II

Remarks

1. $\mathcal{El}_\mathfrak{A}(\mathcal{B}(\mathcal{H}))$ is a subalgebra of $\mathcal{CB}(\mathcal{B}(\mathcal{H}))$.

2. $T \in \mathcal{El}_\mathfrak{A}(\mathcal{B}(\mathcal{H})) \cap CP(\mathcal{B}(\mathcal{H}))$ iff there are $n \in \mathbb{N}$, $c := (c_1, \ldots, c_n) \in \mathfrak{A}^n$ such that $T = M_{c,c^*}$ with $c^* := (c_1^*, \ldots, c_n^*)$. 

Question

Given $T \in \mathcal{El}_\mathfrak{A}(\mathcal{B}(\mathcal{H}))$, can a Wittstock modulus $|T| \in \mathcal{CB}(\mathcal{B}(\mathcal{H}))$ be found in $\mathcal{El}_\mathfrak{A}(\mathcal{B}(\mathcal{H}))$?

Answer

Yes!
Elementary operators, II

Remarks

1. $\mathcal{E}_\mathcal{A}(B(H))$ is a subalgebra of $CB(B(H))$.
2. $T \in \mathcal{E}_\mathcal{A}(B(H)) \cap CP(B(H))$ iff there are $n \in \mathbb{N}$, $c := (c_1, \ldots, c_n) \in \mathcal{A}^n$ such that $T = M_{c,c^*}$ with $c^* := (c_1^*, \ldots, c_n^*)$.
3. $M_{a,b}^* = M_{b^*,a^*}$. 
Remarks

1. $\mathcal{E}_\mathcal{A}(\mathcal{B}(\mathcal{H}))$ is a subalgebra of $\mathcal{C}B(\mathcal{B}(\mathcal{H}))$.

2. $T \in \mathcal{E}_\mathcal{A}(\mathcal{B}(\mathcal{H})) \cap \mathcal{C}P(\mathcal{B}(\mathcal{H}))$ iff there are $n \in \mathbb{N}$, $c := (c_1, \ldots, c_n) \in \mathcal{A}^n$ such that $T = M_{c,c^*}$ with $c^* := (c_1^*, \ldots, c_n^*)$.

3. $M_{a,b}^* = M_{b^*,a^*}$.

4. $\mathcal{E}_\mathcal{A}(\mathcal{B}(\mathcal{H})) \cong \mathcal{A} \otimes_h \mathcal{A}$. 

Given $T \in \mathcal{E}_\mathcal{A}(\mathcal{B}(\mathcal{H}))$, can a Wittstock modulus $|T| \in \mathcal{C}B(\mathcal{B}(\mathcal{H}))$ be found in $\mathcal{E}_\mathcal{A}(\mathcal{B}(\mathcal{H}))$? 

Answer: Yes!
Elementary operators, II

Remarks

1. $\mathcal{E}_{\mathcal{A}}(B(\mathcal{H}))$ is a subalgebra of $CB(B(\mathcal{H}))$.

2. $T \in \mathcal{E}_{\mathcal{A}}(B(\mathcal{H})) \cap CP(B(\mathcal{H}))$ iff there are $n \in \mathbb{N}$, $c := (c_1, \ldots, c_n) \in \mathcal{A}^n$ such that $T = M_{c, c^*}$ with $c^* := (c_1^*, \ldots, c_n^*)$.

3. $M_{a,b}^* = M_{b^*,a^*}$.

4. $\mathcal{E}_{\mathcal{A}}(B(\mathcal{H})) \cong \mathcal{A} \otimes_h \mathcal{A}$. ($\otimes_h = $ Haagerup tensor product)
Remarks

1. $\mathcal{E}_{\mathcal{A}}(\mathcal{B}(\mathcal{H}))$ is a subalgebra of $\mathcal{CB}(\mathcal{B}(\mathcal{H}))$.
2. $T \in \mathcal{E}_{\mathcal{A}}(\mathcal{B}(\mathcal{H})) \cap \mathcal{CP}(\mathcal{B}(\mathcal{H}))$ iff there are $n \in \mathbb{N}$, $c := (c_1, \ldots, c_n) \in \mathcal{A}^n$ such that $T = M_{c,c^*}$ with $c^* := (c_1^*, \ldots, c_n^*)$.
3. $M^*_{a,b} = M_{b^*,a^*}$.
4. $\mathcal{E}_{\mathcal{A}}(\mathcal{B}(\mathcal{H})) \cong \mathcal{A} \otimes_h \mathcal{A}$. ($\otimes_h = \text{Haagerup tensor product}$)
Elementary operators, II

Remarks

1. \( \mathcal{E}_{\mathcal{A}}(B(\mathcal{H})) \) is a subalgebra of \( CB(B(\mathcal{H})) \).
2. \( T \in \mathcal{E}_{\mathcal{A}}(B(\mathcal{H})) \cap CP(B(\mathcal{H})) \) iff there are \( n \in \mathbb{N} \),
\( c := (c_1, \ldots, c_n) \in \mathcal{A}^n \) such that \( T = M_{c,c^*} \) with \( c^* := (c_1^*, \ldots, c_n^*) \).
3. \( M_{a,b}^* = M_{b^*,a^*} \).
4. \( \mathcal{E}_{\mathcal{A}}(B(\mathcal{H})) \cong \mathcal{A} \otimes_h \mathcal{A} \). (\( \otimes_h = \text{Haagerup tensor product} \))

Question

Given \( T \in \mathcal{E}_{\mathcal{A}}(B(\mathcal{H})) \),
Elementary operators, II

Remarks

1. $\ell_{21}(B(H))$ is a subalgebra of $CB(B(H))$.

2. $T \in \ell_{21}(B(H)) \cap CP(B(H))$ iff there are $n \in \mathbb{N}$, $c := (c_1, \ldots, c_n) \in \mathcal{A}^n$ such that $T = M_{c,c^*}$ with $c^* := (c_1^*, \ldots, c_n^*)$.

3. $M_{a,b}^* = M_{b^*,a^*}$.

4. $\ell_{21}(B(H)) \cong \mathcal{A} \otimes_h \mathcal{A}$. ($\otimes_h =$ Haagerup tensor product)

Question

Given $T \in \ell_{21}(B(H))$, can a Wittstock modulus $|T| \in CB(B(H))$ be found.
Elementary operators, II

Remarks

1. $\mathcal{E}_{\mathfrak{A}}(B(\mathcal{H}))$ is a subalgebra of $CB(B(\mathcal{H}))$.

2. $T \in \mathcal{E}_{\mathfrak{A}}(B(\mathcal{H})) \cap \mathcal{CP}(B(\mathcal{H}))$ iff there are $n \in \mathbb{N}$, $c := (c_1, \ldots, c_n) \in \mathfrak{A}^n$ such that $T = M_{c,c^*}$ with $c^* := (c_1^*, \ldots, c_n^*)$.

3. $M_{a,b}^* = M_{b^*,a^*}$.

4. $\mathcal{E}_{\mathfrak{A}}(B(\mathcal{H})) \cong \mathfrak{A} \otimes_h \mathfrak{A}$. ($\otimes_h =$ Haagerup tensor product)

Question

Given $T \in \mathcal{E}_{\mathfrak{A}}(B(\mathcal{H}))$, can a Wittstock modulus $|T| \in CB(B(\mathcal{H}))$ be found in $\mathcal{E}_{\mathfrak{A}}(B(\mathcal{H}))$?
Remarks

1. $\mathcal{E}_\mathfrak{A}(\mathcal{B}(\mathcal{H}))$ is a subalgebra of $\mathcal{C}\mathcal{B}(\mathcal{B}(\mathcal{H}))$.
2. $T \in \mathcal{E}_\mathfrak{A}(\mathcal{B}(\mathcal{H})) \cap \mathcal{C}\mathcal{P}(\mathcal{B}(\mathcal{H}))$ iff there are $n \in \mathbb{N}$, $c := (c_1, \ldots, c_n) \in \mathfrak{A}^n$ such that $T = M_{c,c^*}$ with $c^* := (c_1^*, \ldots, c_n^*)$.
3. $M_{a,b}^* = M_{b^*,a^*}$.
4. $\mathcal{E}_\mathfrak{A}(\mathcal{B}(\mathcal{H})) \cong \mathfrak{A} \otimes_h \mathfrak{A}$. ($\otimes_h = \text{Haagerup tensor product}$)

Question

Given $T \in \mathcal{E}_\mathfrak{A}(\mathcal{B}(\mathcal{H}))$, can a Wittstock modulus $|T| \in \mathcal{C}\mathcal{B}(\mathcal{B}(\mathcal{H}))$ be found in $\mathcal{E}_\mathfrak{A}(\mathcal{B}(\mathcal{H}))$?

Answer

Yes!
Elementary operators, II

Remarks

1. $\mathcal{E}_{A}(B(\mathcal{H}))$ is a subalgebra of $CB(B(\mathcal{H}))$.
2. $T \in \mathcal{E}_{A}(B(\mathcal{H})) \cap CP(B(\mathcal{H}))$ iff there are $n \in \mathbb{N}$, $c := (c_1, \ldots, c_n) \in A^n$ such that $T = M_{c,c^*}$ with $c^* := (c_1^*, \ldots, c_n^*)$.
3. $M_{a,b}^* = M_{b^*,a^*}$.
4. $\mathcal{E}_{A}(B(\mathcal{H})) \cong A \otimes \text{h} A$. ($\otimes \text{h} = $ Haagerup tensor product)

Question

Given $T \in \mathcal{E}_{A}(B(\mathcal{H}))$, can a Wittstock modulus $|T| \in CB(B(\mathcal{H}))$ be found in $\mathcal{E}_{A}(B(\mathcal{H}))$?

Answer

Yes!
Let \( T \in \mathbb{E}_A^\ell(B(H)) \). Then there are \( n \in \mathbb{N}, a, b \in A^n \) such that
\[
T = M_{a, a^*} + \cdots + M_{a^n, a_{n}^*}
\]
and
\[
\|T\|_{cb} = \|a_1 a^* + \cdots + a_n a_{n}^*\|.
\]
Then \( |T| = \frac{1}{2} M_{a, a^*} + \frac{1}{2} M_{b^*, b} \) is a Wittstock modulus for \( T \).

If \( T(id_H) = id_H \), then
\[
M_{a, a^*}(id_H) = M_{b^*, b}(id_H) = \|T\|_{cb} id_H.
\]
Example

Let $T \in E_{\ell} A(B(H))$. Then there are $n \in \mathbb{N}$, $a, b \in A$ such that $T = M_{a, b}$ and

$$
\|T\|_{cb} = \|a_1^*a + \cdots + a_n^*a_n\| = \|b_1^*b + \cdots + b_n^*b_n\|.
$$

Then $|T| := \frac{1}{2}M_{a, a^*} + \frac{1}{2}M_{b^*, b}$ is a Wittstock modulus for $T$.

If $T(id_H) = id_H$, then $M_{a, a^*(id_H)} = M_{b^*, b(id_H)} = \|T\|_{cb id_H}$. 

Advertisements
Example

Let \( T \in \mathcal{E}_{\ell_2}(\mathcal{B}(H)) \).
Example

Let \( T \in \mathcal{E}_{\mathcal{A}}(\mathcal{B}(H)) \). Then there are
Example

Let $T \in \mathcal{E}_{\mathbb{A}}(\mathcal{B}(\mathcal{H}))$. Then there are $n \in \mathbb{N}$,
Example

Let $T \in \mathcal{E}_{\mathcal{A}}(\mathcal{B}(\mathcal{H}))$. Then there are $n \in \mathbb{N}$, $a, b \in \mathcal{A}^n$ such that $T = M_{a, a^*} = M_{b^*, b}$ and

$$
\|T\| = \|a_1 a^* + \cdots + a_n a^*\| = \|b^* b + \cdots + b_n b^*\|.
$$

Then $|T| = \frac{1}{2} M_{a, a^*} + \frac{1}{2} M_{b^*, b}$ is a Wittstock modulus for $T$. If $T(id_{\mathcal{H}}) = id_{\mathcal{H}}$, then $M_{a, a^*}(id_{\mathcal{H}}) = M_{b^*, b}(id_{\mathcal{H}}) = \|T\| c_{id_{\mathcal{H}}}$.
Elementary operators, III

Example

Let $T \in \mathcal{E}_{\mathcal{A}}(\mathcal{B}(\mathcal{H}))$. Then there are $n \in \mathbb{N}$, $a, b \in \mathcal{A}^n$ such that

$$T = M_{a,b}$$
Example

Let \( T \in \mathcal{E}_{\mathfrak{A}}(\mathcal{B}(\mathcal{H})) \). Then there are \( n \in \mathbb{N}, \ a, b \in \mathfrak{A}^n \) such that

\[
T = M_{a,b}
\]

and
Elementary operators, III

Example

Let $T \in \mathcal{E}_{\mathcal{A}}(\mathcal{B}(\mathcal{H}))$. Then there are $n \in \mathbb{N}$, $a, b \in \mathbb{A}^n$ such that

$$T = M_{a,b}$$

and

$$\| T \|_{cb} = \| a_1a^* + \cdots a_na_n^* \| = \| b_1^*b_1 + \cdots + b_n^*b_n \|.$$
Example

Let $T \in \mathcal{E}_{\mathcal{A}}(\mathcal{B}(\mathcal{H}))$. Then there are $n \in \mathbb{N}$, $a, b \in \mathcal{A}^n$ such that

$$T = M_{a,b}$$

and

$$\| T \|_{cb} = \| a_1 a^* + \cdots + a_n a_n^* \| = \| b_1^* b_1 + \cdots + b_n^* b_n \|.$$

Then

$$| T | := \frac{1}{2} M_{a,a^*} + \frac{1}{2} M_{b^*,b}.$$
Example

Let $T \in \mathcal{E}_{\mathcal{A}}(\mathcal{B}(\mathcal{H}))$. Then there are $n \in \mathbb{N}$, $a, b \in \mathcal{A}^n$ such that

$$T = M_{a, b}$$

and

$$\|T\|_{cb} = \|a_1 a^* + \cdots + a_n a_n^*\| = \|b_1^* b_1 + \cdots + b_n^* b_n\|.$$ 

Then

$$|T| := \frac{1}{2} M_{a, a^*} + \frac{1}{2} M_{b^*, b}$$

is a Wittstock modulus for $T$. 
Example

Let \( T \in \mathcal{E}_\mathfrak{A}(\mathcal{B}(\mathcal{H})) \). Then there are \( n \in \mathbb{N}, \ a, b \in \mathfrak{A}^n \) such that

\[
T = M_{a,b}
\]

and

\[
\| T \|_{cb} = \| a_1 a^* + \cdots a_n a_n^* \| = \| b_1^* b_1 + \cdots + b_n^* b_n \|.\]

Then

\[
| T | := \frac{1}{2} M_{a,a^*} + \frac{1}{2} M_{b^*,b}
\]

is a Wittstock modulus for \( T \). If \( T(\text{id}_{\mathcal{H}}) = \text{id}_{\mathcal{H}} \),
Example

Let \( T \in \mathcal{E}_{\mathfrak{A}}(\mathcal{B}(\mathcal{H})) \). Then there are \( n \in \mathbb{N} \), \( a, b \in \mathfrak{A}^n \) such that

\[
T = M_{a,b}
\]

and

\[
\| T \|_{cb} = \| a_1 a^* + \cdots + a_n a^*_n \| = \| b_1^* b_1 + \cdots + b_n^* b_n \|.
\]

Then

\[
| T | := \frac{1}{2} M_{a,a^*} + \frac{1}{2} M_{b^*,b}
\]

is a Wittstock modulus for \( T \). If \( T(\text{id}_{\mathcal{H}}) = \text{id}_{\mathcal{H}} \), then

\[
M_{a,a^*}(\text{id}_{\mathcal{H}}) = M_{b^*,b}(\text{id}_{\mathcal{H}}) = \| T \|_{cb} \text{id}_{\mathcal{H}}.
\]
Remark

Let $A$ be a Banach algebra, and let $E$ be a Banach $A$-bimodule. Then $E^*$ becomes a Banach $A$-bimodule via

$\langle a \cdot \phi, x \rangle := \langle \phi, x \cdot a \rangle$

and

$\langle \phi \cdot a, x \rangle := \langle \phi, a \cdot x \rangle$

for $a \in A$, $x \in E$, $\phi \in E^*$.

Definition (Johnson, 1972)

A Banach algebra $A$ is called amenable if, for every Banach $A$-bimodule $E$, every derivation $D: A \to E^*$ is inner, i.e., there is $\phi \in E^*$ such that $Dx = \text{ad} \phi x := x \cdot \phi - \phi \cdot x$ ($x \in A$).
### Remark

Let $A$ be a Banach algebra, and let $E$ be a Banach $A$-bimodule. Then $E^*$ becomes a Banach $A$-bimodule via \[ \langle a \cdot \varphi, x \rangle := \langle \varphi, x \cdot a \rangle \] and \[ \langle \varphi \cdot a, x \rangle := \langle \varphi, a \cdot x \rangle \] for $a \in A$, $x \in E$, $\varphi \in E^*$. 

**Definition (Johnson, 1972)**

A Banach algebra $A$ is called amenable if, for every Banach $A$-bimodule $E$, every derivation $D : A \to E^*$ is inner, i.e., there is $\varphi \in E^*$ such that $Dx = \text{ad}_\varphi x := x \cdot \varphi - \varphi \cdot x$ ($x \in A$).
Remark

Let $\mathcal{A}$ be a Banach algebra,
Amenable Banach algebras, I

Remark

Let $\mathcal{A}$ be a Banach algebra, and let $E$ be a Banach $\mathcal{A}$-bimodule.
Remark

Let $A$ be a Banach algebra, and let $E$ be a Banach $A$-bimodule. Then $E^*$ becomes a Banach $A$-bimodule.
## Remark

Let $\mathcal{A}$ be a Banach algebra, and let $E$ be a Banach $\mathcal{A}$-bimodule. Then $E^*$ becomes a Banach $\mathcal{A}$-bimodule via

$$\langle a \cdot \phi, x \rangle := \langle \phi, x \cdot a \rangle \quad \text{and} \quad \langle \phi \cdot a, x \rangle := \langle \phi, a \cdot x \rangle$$
Remark

Let $\mathcal{A}$ be a Banach algebra, and let $E$ be a Banach $\mathcal{A}$-bimodule. Then $E^*$ becomes a Banach $\mathcal{A}$-bimodule via

$$\langle a \cdot \phi, x \rangle := \langle \phi, x \cdot a \rangle \quad \text{and} \quad \langle \phi \cdot a, x \rangle := \langle \phi, a \cdot x \rangle$$

for
Remark

Let $\mathcal{A}$ be a Banach algebra, and let $E$ be a Banach $\mathcal{A}$-bimodule. Then $E^*$ becomes a Banach $\mathcal{A}$-bimodule via

$$\langle a \cdot \phi, x \rangle := \langle \phi, x \cdot a \rangle \quad \text{and} \quad \langle \phi \cdot a, x \rangle := \langle \phi, a \cdot x \rangle$$

for $a \in \mathcal{A}$,
Remark

Let $\mathcal{A}$ be a Banach algebra, and let $E$ be a Banach $\mathcal{A}$-bimodule. Then $E^*$ becomes a Banach $\mathcal{A}$-bimodule via

$$\langle a \cdot \phi, x \rangle := \langle \phi, x \cdot a \rangle \quad \text{and} \quad \langle \phi \cdot a, x \rangle := \langle \phi, a \cdot x \rangle$$

for $a \in \mathcal{A}$, $x \in E$,.
Amenable Banach algebras, I

Remark

Let $\mathcal{A}$ be a Banach algebra, and let $E$ be a Banach $\mathcal{A}$-bimodule. Then $E^*$ becomes a Banach $\mathcal{A}$-bimodule via

$$\langle a \cdot \phi, x \rangle := \langle \phi, x \cdot a \rangle \quad \text{and} \quad \langle \phi \cdot a, x \rangle := \langle \phi, a \cdot x \rangle$$

for $a \in \mathcal{A}$, $x \in E$, $\phi \in E^*$. 
Remark

Let $\mathcal{A}$ be a Banach algebra, and let $E$ be a Banach $\mathcal{A}$-bimodule. Then $E^*$ becomes a Banach $\mathcal{A}$-bimodule via

$$\langle a \cdot \phi, x \rangle := \langle \phi, x \cdot a \rangle \quad \text{and} \quad \langle \phi \cdot a, x \rangle := \langle \phi, a \cdot x \rangle$$

for $a \in \mathcal{A}$, $x \in E$, $\phi \in E^*$. 

Definition (Johnson, 1972)

A Banach algebra $\mathcal{A}$ is called amenable if, for every Banach $\mathcal{A}$-bimodule $E$, every derivation $D: \mathcal{A} \to E^*$ is inner, i.e., there is $\phi \in E^*$ such that $Dx = \text{ad}_\phi x := x \cdot \phi - \phi \cdot x$ ($x \in A$).
Remark

Let \( \mathcal{A} \) be a Banach algebra, and let \( E \) be a Banach \( \mathcal{A} \)-bimodule. Then \( E^* \) becomes a Banach \( \mathcal{A} \)-bimodule via

\[
\langle a \cdot \phi, x \rangle := \langle \phi, x \cdot a \rangle \quad \text{and} \quad \langle \phi \cdot a, x \rangle := \langle \phi, a \cdot x \rangle
\]

for \( a \in \mathcal{A} \), \( x \in E \), \( \phi \in E^* \).

Definition (Johnson, 1972)

A Banach algebra \( \mathcal{A} \)
Amenable Banach algebras, I

Remark

Let $\mathcal{A}$ be a Banach algebra, and let $E$ be a Banach $\mathcal{A}$-bimodule. Then $E^*$ becomes a Banach $\mathcal{A}$-bimodule via

$$\langle a \cdot \phi, x \rangle := \langle \phi, x \cdot a \rangle \quad \text{and} \quad \langle \phi \cdot a, x \rangle := \langle \phi, a \cdot x \rangle$$

for $a \in \mathcal{A}, \ x \in E, \ \phi \in E^*.$

Definition (Johnson, 1972)

A Banach algebra $\mathcal{A}$ is called amenable
Amenable Banach algebras, I

**Remark**

Let $\mathfrak{A}$ be a Banach algebra, and let $E$ be a Banach $\mathfrak{A}$-bimodule. Then $E^*$ becomes a Banach $\mathfrak{A}$-bimodule via

$$\langle a \cdot \phi, x \rangle := \langle \phi, x \cdot a \rangle \quad \text{and} \quad \langle \phi \cdot a, x \rangle := \langle \phi, a \cdot x \rangle$$

for $a \in \mathfrak{A}$, $x \in E$, $\phi \in E^*$.

**Definition (Johnson, 1972)**

A Banach algebra $\mathfrak{A}$ is called **amenable** if,
### Remark

Let \( \mathcal{A} \) be a Banach algebra, and let \( E \) be a Banach \( \mathcal{A} \)-bimodule. Then \( E^* \) becomes a Banach \( \mathcal{A} \)-bimodule via

\[
\langle a \cdot \phi, x \rangle := \langle \phi, x \cdot a \rangle \quad \text{and} \quad \langle \phi \cdot a, x \rangle := \langle \phi, a \cdot x \rangle
\]

for \( a \in \mathcal{A}, \ x \in E, \ \phi \in E^* \).

### Definition (Johnson, 1972)

A Banach algebra \( \mathcal{A} \) is called **amenable** if, for every Banach \( \mathcal{A} \)-bimodule \( E \),
Remark

Let $\mathcal{A}$ be a Banach algebra, and let $E$ be a Banach $\mathcal{A}$-bimodule. Then $E^*$ becomes a Banach $\mathcal{A}$-bimodule via

$$\langle a \cdot \phi, x \rangle := \langle \phi, x \cdot a \rangle \quad \text{and} \quad \langle \phi \cdot a, x \rangle := \langle \phi, a \cdot x \rangle$$

for $a \in \mathcal{A}$, $x \in E$, $\phi \in E^*$.

Definition (Johnson, 1972)

A Banach algebra $\mathcal{A}$ is called amenable if, for every Banach $\mathcal{A}$-bimodule $E$, every derivation $D : \mathcal{A} \to E^*$
Remark

Let $\mathcal{A}$ be a Banach algebra, and let $E$ be a Banach $\mathcal{A}$-bimodule. Then $E^*$ becomes a Banach $\mathcal{A}$-bimodule via

$$\langle a \cdot \phi, x \rangle := \langle \phi, x \cdot a \rangle \quad \text{and} \quad \langle \phi \cdot a, x \rangle := \langle \phi, a \cdot x \rangle$$

for $a \in \mathcal{A}, \ x \in E, \ \phi \in E^*$.

Definition (Johnson, 1972)

A Banach algebra $\mathcal{A}$ is called \textit{amenable} if, for every Banach $\mathcal{A}$-bimodule $E$, every derivation $D : \mathcal{A} \to E^*$ is \textit{inner},
Remark

Let $\mathcal{A}$ be a Banach algebra, and let $E$ be a Banach $\mathcal{A}$-bimodule. Then $E^*$ becomes a Banach $\mathcal{A}$-bimodule via

$$\langle a \cdot \phi, x \rangle := \langle \phi, x \cdot a \rangle$$

and

$$\langle \phi \cdot a, x \rangle := \langle \phi, a \cdot x \rangle$$

for $a \in \mathcal{A}$, $x \in E$, $\phi \in E^*$.

Definition (Johnson, 1972)

A Banach algebra $\mathcal{A}$ is called amenable if, for every Banach $\mathcal{A}$-bimodule $E$, every derivation $D : \mathcal{A} \to E^*$ is inner, i.e., there is $\phi \in E^*$ such that

$$Dx = \text{ad}_\phi x := x \cdot \phi - \phi \cdot x \quad (x \in \mathcal{A}).$$
Amenable Banach algebras, II

Wittstock moduli of elementary operators and their application to generalized notions of amenability

Volker Runde

Wittstock moduli of elementary operators

Generalized notions of amenability

Paulsen’s off-diagonal technique

Conclusion

Advertisements

Amenable Banach algebras, II

Definition: A net \((d_\alpha)\) \(\alpha \in \mathbb{A} \otimes_\gamma \mathbb{A}\) is called an approximate diagonal for \(\mathbb{A}\) if \(a \cdot d_\alpha - d_\alpha \cdot a \to 0\) \((a \in \mathbb{A})\) and \(a \Delta d_\alpha \to a\) \((a \in \mathbb{A})\).
Amenable Banach algebras, II

Notation
Notation

⊗\gamma:

completed Banach space tensor product.

\Delta: the multiplication map

A ⊗ γ A \ni a ⊗ b \mapsto ab.

Definition

A net \((d_\alpha)_{\alpha} \subset A \otimes \gamma A\) is called an approximate diagonal for \(A\) if

\[a \cdot d_\alpha - d_\alpha \cdot a \to 0 (a \in A)\]

and

\[a \Delta d_\alpha \to a (a \in A)\].
Notation

⊗γ: completed Banach space tensor product.
Amenable Banach algebras, II

Notation

⊗γ: completed Banach space tensor product.

∆:
Amenable Banach algebras, II

Notation

$\otimes^\gamma$: completed Banach space tensor product.  
$\Delta$: the multiplication map $A \otimes^\gamma A \ni a \otimes b \mapsto ab$. 
Amenable Banach algebras, II

Notation

$\otimes^\gamma$: completed Banach space tensor product.

$\Delta$: the multiplication map $A \otimes^\gamma A \ni a \otimes b \mapsto ab$. 

Definition
Amenable Banach algebras, II

Notation

⊗γ: completed Banach space tensor product.
∆: the multiplication map \( A ⊗ γ A \ni a ⊗ b \mapsto ab \).

Definition

A net \((d_\alpha)_{\alpha} \subset A ⊗ γ A\).
Amenable Banach algebras, II

**Notation**

\( \otimes^\gamma \): completed Banach space tensor product.

\( \Delta : \) the multiplication map \( A \otimes^\gamma A \ni a \otimes b \mapsto ab \).

**Definition**

A net \( (d_\alpha)_{\alpha} \subset A \otimes^\gamma A \) is called an approximate diagonal for \( A \).
Amenable Banach algebras, II

Notation

⊗γ: completed Banach space tensor product.

Δ: the multiplication map A ⊗γ A ⊇ a ⊗ b ↦→ ab.

Definition

A net \((d_\alpha)_\alpha \subset A \otimes_\gamma A\) is called an approximate diagonal for A if

\[ a \cdot d_\alpha - d_\alpha \cdot a \to 0 \quad (a \in A) \]
Amenable Banach algebras, II

Notation

$\otimes^\gamma$: completed Banach space tensor product.
$\Delta$: the multiplication map $A \otimes^\gamma A \ni a \otimes b \mapsto ab$.

Definition

A net $(d_\alpha)_\alpha \subset A \otimes^\gamma A$ is called an approximate diagonal for $A$ if

$$a \cdot d_\alpha - d_\alpha \cdot a \to 0 \quad (a \in A)$$

and

$$a \Delta d_\alpha \to a \quad (a \in A).$$
Amenable Banach algebras, III

Theorem (Johnson, 1972)

The following are equivalent for a Banach algebra $A$:

1. $A$ is amenable;
2. $A$ has a bounded approximate diagonal.

Theorem (Johnson, 1972)

The following are equivalent for a locally compact group $G$:

1. $L_1(G)$ is amenable;
2. $G$ is amenable.
Amenable Banach algebras, III

Theorem (Johnson, 1972)

The following are equivalent for a Banach algebra $A$:
1. $A$ is amenable;
2. $A$ has a bounded approximate diagonal.

Theorem (Johnson, 1972)

The following are equivalent for a locally compact group $G$:
1. $L^1(G)$ is amenable;
2. $G$ is amenable.
Theorem (Johnson, 1972)

The following are equivalent for a Banach algebra $\mathcal{A}$:
Theorem (Johnson, 1972)

The following are equivalent for a Banach algebra $\mathcal{A}$:

1. $\mathcal{A}$ is amenable;
Theorem (Johnson, 1972)

The following are equivalent for a Banach algebra $\mathbb{A}$:

1. $\mathbb{A}$ is amenable;
2. $\mathbb{A}$ has a **bounded** approximate diagonal.
Amenable Banach algebras, III

Theorem (Johnson, 1972)

The following are equivalent for a Banach algebra \( \mathbf{A} \):

1. \( \mathbf{A} \) is amenable;
2. \( \mathbf{A} \) has a \textit{bounded} approximate diagonal.

Theorem (Johnson, 1972)

The following are equivalent for a locally compact group \( G \):

1. \( L^1(G) \) is amenable;
2. \( G \) is amenable.
Theorem (Johnson, 1972)

The following are equivalent for a Banach algebra $\mathcal{A}$:

1. $\mathcal{A}$ is amenable;
2. $\mathcal{A}$ has a \textit{bounded} approximate diagonal.

Theorem (Johnson, 1972)

The following are equivalent for a locally compact group $G$:
Amenable Banach algebras, III

Theorem (Johnson, 1972)

The following are equivalent for a Banach algebra $\mathcal{A}$:

1. $\mathcal{A}$ is amenable;
2. $\mathcal{A}$ has a bounded approximate diagonal.

Theorem (Johnson, 1972)

The following are equivalent for a locally compact group $G$:

1. $L^1(G)$ is amenable;
Amenable Banach algebras, III

Theorem (Johnson, 1972)

The following are equivalent for a Banach algebra $\mathfrak{A}$:

1. $\mathfrak{A}$ is amenable;
2. $\mathfrak{A}$ has a \textit{bounded} approximate diagonal.

Theorem (Johnson, 1972)

The following are equivalent for a locally compact group $G$:

1. $L^1(G)$ is amenable;
2. $G$ is amenable.
Approximately amenable Banach algebras

Definition (Ghahramani–Loy, 2004)
A Banach algebra $A$ is called approximately amenable if, for every Banach $A$-bimodule $E$, every derivation $D: A \to E^*$ is approximately inner, i.e., there is a net $(\phi_\alpha)_{\alpha \in \mathcal{A}} \subset E^*$ such that $Dx = \lim_{\alpha} \text{ad} \phi_\alpha x$ ($x \in A$).

Example
Let $G$ be a finite, non-abelian group. Then $c_0 \bigoplus_{n=1}^\infty A(G^n)$ is approximately amenable, but not amenable.
Definition (Ghahramani–Loy, 2004)

A Banach algebra \( A \) is called approximately amenable if, for every Banach \( A \)-bimodule \( E \), every derivation \( D: A \to E^* \) is approximately inner, i.e., there is a net \((\phi_\alpha)_{\alpha} \subset E^*\) such that \( Dx = \lim_{\alpha} \text{ad} \phi_\alpha x \) \((x \in A)\).

Example: Let \( G \) be a finite, non-abelian group. Then \( c_0 \bigoplus_{n=1}^{\infty} A(G_n) \) is approximately amenable, but not amenable.
Approximately amenable Banach algebras

**Definition (Ghahramani–Loy, 2004)**

A Banach algebra $\mathcal{A}$

**Definition (Ghahramani–Loy, 2004)**

A Banach algebra $\mathcal{A}$ is called approximately amenable if, for every Banach $\mathcal{A}$-bimodule $E$, every derivation $D: \mathcal{A} \to E^*$ is approximately inner, i.e., there is a net $(\phi_\alpha)_{\alpha \in \mathbb{A}} \subset E^*$ such that $Dx = \lim_{\alpha} \text{ad} \phi_\alpha x$ ($x \in \mathcal{A}$).

**Example**

Let $G$ be a finite, non-abelian group. Then $c_0 \bigoplus_{n=1}^\infty \mathcal{A}(G^n)$ is approximately amenable, but not amenable.
Approximately amenable Banach algebras

Definition (Ghahramani–Loy, 2004)

A Banach algebra $\mathcal{A}$ is called **approximately amenable**

A Banach algebra $\mathcal{A}$ is called approximately amenable.
Approximately amenable Banach algebras

**Definition (Ghahramani–Loy, 2004)**

A Banach algebra \( \mathcal{A} \) is called **approximately amenable** if,

\[
\text{for every Banach } \mathcal{A} \text{-bimodule } E, \text{ every derivation } D: \mathcal{A} \to E^* \text{ is approximately inner,} \]

i.e., there is a net \( (\phi_\alpha)_{\alpha} \subset E^* \) such that

\[
Dx = \lim_\alpha \text{ad} \phi_\alpha x \quad (x \in \mathcal{A}).
\]

**Example**

Let \( \mathcal{G} \) be a finite, non-abelian group. Then

\[
\ell^\infty \bigoplus_{n=1}^\infty \mathcal{A}(\mathcal{G}^n)
\]

is approximately amenable, but not amenable.
Definition (Ghahramani–Loy, 2004)

A Banach algebra $\mathfrak{A}$ is called approximately amenable if, for every Banach $\mathfrak{A}$-bimodule $E$,
Definition (Ghahramani–Loy, 2004)

A Banach algebra \( \mathfrak{A} \) is called \textbf{approximately amenable} if, for every Banach \( \mathfrak{A} \)-bimodule \( E \), every derivation \( D: \mathfrak{A} \to E^* \)
Approximately amenable Banach algebras

Definition (Ghahramani–Loy, 2004)

A Banach algebra $\mathcal{A}$ is called **approximately amenable** if, for every Banach $\mathcal{A}$-bimodule $E$, every derivation $D: \mathcal{A} \rightarrow E^*$ is **approximately inner**, 

$\text{(Example)}$ Let $G$ be a finite, non-abelian group. Then $c_0^\infty \bigoplus_{n=1}^\infty A(G^n)$ is approximately amenable, but not amenable.
Definition (Ghahramani–Loy, 2004)

A Banach algebra $\mathfrak{A}$ is called approximately amenable if, for every Banach $\mathfrak{A}$-bimodule $E$, every derivation $D: \mathfrak{A} \rightarrow E^*$ is approximately inner, i.e., there is a net $(\phi_\alpha)_\alpha \subseteq E^*$.
Definition (Ghahramani–Loy, 2004)

A Banach algebra $\mathcal{A}$ is called **approximately amenable** if, for every Banach $\mathcal{A}$-bimodule $E$, every derivation $D: \mathcal{A} \to E^*$ is **approximately inner**, i.e., there is a net $(\phi_\alpha)_{\alpha} \subset E^*$ such that

$$Dx = \lim_{\alpha} \text{ad}_{\phi_\alpha} x \quad (x \in \mathcal{A}).$$
Approximately amenable Banach algebras

Definition (Ghahramani–Loy, 2004)

A Banach algebra $\mathcal{A}$ is called approximately amenable if, for every Banach $\mathcal{A}$-bimodule $E$, every derivation $D: \mathcal{A} \rightarrow E^*$ is approximately inner, i.e., there is a net $(\phi_\alpha)_\alpha \subset E^*$ such that

$$Dx = \lim_{\alpha} \text{ad}_{\phi_\alpha} x \quad (x \in \mathcal{A}).$$

Example
Definition (Ghahramani–Loy, 2004)

A Banach algebra $\mathfrak{A}$ is called approximately amenable if, for every Banach $\mathfrak{A}$-bimodule $E$, every derivation $D: \mathfrak{A} \to E^*$ is approximately inner, i.e., there is a net $(\phi_\alpha)_\alpha \subset E^*$ such that

$$Dx = \lim_{\alpha} \text{ad}_{\phi_\alpha} x \quad (x \in \mathfrak{A}).$$

Example

Let $G$ be a finite,
Approximately amenable Banach algebras

Definition (Ghahramani–Loy, 2004)

A Banach algebra $\mathfrak{A}$ is called approximately amenable if, for every Banach $\mathfrak{A}$-bimodule $E$, every derivation $D: \mathfrak{A} \rightarrow E^*$ is approximately inner, i.e., there is a net $(\phi_\alpha)_{\alpha} \subset E^*$ such that

$$Dx = \lim_{\alpha} \text{ad}_{\phi_\alpha} x \quad (x \in \mathfrak{A}).$$

Example

Let $G$ be a finite, non-abelian group.
Approximately amenable Banach algebras

**Definition (Ghahramani–Loy, 2004)**

A Banach algebra $\mathcal{A}$ is called **approximately amenable** if, for every Banach $\mathcal{A}$-bimodule $E$, every derivation $D: \mathcal{A} \rightarrow E^*$ is **approximately inner**, i.e., there is a net $(\phi_\alpha) \subset E^*$ such that

$$Dx = \lim_{\alpha} \text{ad}_{\phi_\alpha} x \quad (x \in \mathcal{A}).$$

**Example**

Let $G$ be a finite, non-abelian group. Then

$$c_0 - \bigoplus_{n=1}^{\infty} A(G^n)$$
Definition (Ghahramani–Loy, 2004)

A Banach algebra $\mathcal{A}$ is called approximately amenable if, for every Banach $\mathcal{A}$-bimodule $E$, every derivation $D : \mathcal{A} \to E^*$ is approximately inner, i.e., there is a net $(\phi_\alpha)_\alpha \subset E^*$ such that

$$Dx = \lim_{\alpha} \text{ad}_{\phi_\alpha} x \quad (x \in \mathcal{A}).$$

Example

Let $G$ be a finite, non-abelian group. Then

$$\ell^0 - \bigoplus_{n=1}^{\infty} A(G^n)$$

is approximately amenable,
Approximately amenable Banach algebras

**Definition (Ghahramani–Loy, 2004)**

A Banach algebra $\mathcal{A}$ is called **approximately amenable** if, for every Banach $\mathcal{A}$-bimodule $E$, every derivation $D: \mathcal{A} \to E^*$ is approximately inner, i.e., there is a net $(\phi_\alpha)_{\alpha} \subset E^*$ such that

$$Dx = \lim_{\alpha} \text{ad}\phi_\alpha x \quad (x \in \mathcal{A}).$$

**Example**

Let $G$ be a finite, non-abelian group. Then

$$c_0 - \bigoplus_{n=1}^{\infty} A(G^n)$$

is approximately amenable, but not amenable.
Pseudo-amenable Banach algebras

Definition (Ghahramani–Zhang, 2007)
A Banach algebra $A$ is called pseudo-amenable if it has an approximate diagonal (possibly unbounded).

Example $\ell^p$ for $p \in [1, \infty)$ and $A(F_2)$ are pseudo-amenable, but not approximately amenable.

Proposition (Ghahramani–Zhang, 2007)
The following are equivalent for a Banach algebra $A$ with a BAI:
1. $A$ is pseudo-amenable;
2. $A$ is approximately amenable.
Definition (Ghahramani–Zhang, 2007)

A Banach algebra $A$ is called pseudo-amenable if it has an approximate diagonal (possibly unbounded).

Example $\ell^p$ for $p \in [1, \infty)$ and $A(F_2)$ are pseudo-amenable, but not approximately amenable.

Proposition (Ghahramani–Zhang, 2007)
The following are equivalent for a Banach algebra $A$ with a BAI:
1. $A$ is pseudo-amenable;
2. $A$ is approximately amenable.
Definition (Ghahramani–Zhang, 2007)

A Banach algebra $A$
Definition (Ghahramani–Zhang, 2007)

A Banach algebra $\mathfrak{A}$ is called pseudo-amenable.
**Definition (Ghahramani–Zhang, 2007)**

A Banach algebra $\mathfrak{A}$ is called **pseudo-amenable** if it has an approximate diagonal.
Definition (Ghahramani–Zhang, 2007)

A Banach algebra $\mathfrak{A}$ is called pseudo-amenable if it has an approximate diagonal (possibly unbounded).
Definition (Ghahramani–Zhang, 2007)

A Banach algebra $\mathcal{A}$ is called **pseudo-amenable** if it has an approximate diagonal (possibly unbounded).

Example

$\ell^p$
Definition (Ghahramani–Zhang, 2007)

A Banach algebra $\mathcal{A}$ is called \textbf{pseudo-amenable} if it has an approximate diagonal (possibly unbounded).

Example

$\ell^p$ for $p \in [1, \infty)$
Pseudo-amenable Banach algebras

Definition (Ghahramani–Zhang, 2007)

A Banach algebra $\mathfrak{A}$ is called **pseudo-amenable** if it has an approximate diagonal (possibly unbounded).

Example

$\ell^p$ for $p \in [1, \infty)$ and $A(\mathbb{F}_2)$
Pseudo-amenable Banach algebras

Definition (Ghahramani–Zhang, 2007)

A Banach algebra $\mathfrak{A}$ is called \textbf{pseudo-amenable} if it has an approximate diagonal (possibly unbounded).

Example

$\ell^p$ for $p \in [1, \infty)$ and $A(\mathbb{F}_2)$ are pseudo-amenable,
Definition (Ghahramani–Zhang, 2007)

A Banach algebra $\mathcal{A}$ is called pseudo-amenable if it has an approximate diagonal (possibly unbounded).

Example

$\ell^p$ for $p \in [1, \infty)$ and $A(\mathbb{F}_2)$ are pseudo-amenable, but not approximately amenable.
Pseudo-amenable Banach algebras

Definition (Ghahramani–Zhang, 2007)
A Banach algebra $\mathcal{A}$ is called pseudo-amenable if it has an approximate diagonal (possibly unbounded).

Example
$\ell^p$ for $p \in [1, \infty)$ and $A(\mathbb{F}_2)$ are pseudo-amenable, but not approximately amenable.

Proposition (Ghahramani–Zhang, 2007)
Pseudo-amenable Banach algebras

Definition (Ghahramani–Zhang, 2007)

A Banach algebra $\mathfrak{A}$ is called pseudo-amenable if it has an approximate diagonal (possibly unbounded).

Example

$\ell^p$ for $p \in [1, \infty)$ and $A(\mathbb{F}_2)$ are pseudo-amenable, but not approximately amenable.

Proposition (Ghahramani–Zhang, 2007)

The following are equivalent for a Banach algebra $\mathfrak{A}$
Pseudo-amenable Banach algebras

Definition (Ghahramani–Zhang, 2007)
A Banach algebra $\mathfrak{A}$ is called **pseudo-amenable** if it has an approximate diagonal (possibly unbounded).

Example
$\ell^p$ for $p \in [1, \infty)$ and $A(\mathbb{F}_2)$ are pseudo-amenable, but not approximately amenable.

Proposition (Ghahramani–Zhang, 2007)
*The following are equivalent for a Banach algebra $\mathfrak{A}$ with a BAI:*

- $\mathfrak{A}$ is pseudo-amenable;
- $\mathfrak{A}$ is approximately amenable.
Pseudo-amenable Banach algebras

Definition (Ghahramani–Zhang, 2007)

A Banach algebra $\mathcal{A}$ is called **pseudo-amenable** if it has an approximate diagonal (possibly unbounded).

Example

$L^p$ for $p \in [1, \infty)$ and $A(\mathbb{F}_2)$ are pseudo-amenable, but not approximately amenable.

Proposition (Ghahramani–Zhang, 2007)

The following are equivalent for a Banach algebra $\mathcal{A}$ **with a BAI**:

1. $\mathcal{A}$ is pseudo-amenable;
Pseudo-amenable Banach algebras

Definition (Ghahramani–Zhang, 2007)
A Banach algebra $\mathbb{A}$ is called pseudo-amenable if it has an approximate diagonal (possibly unbounded).

Example
$\ell^p$ for $p \in [1, \infty)$ and $A(\mathbb{F}_2)$ are pseudo-amenable, but not approximately amenable.

Proposition (Ghahramani–Zhang, 2007)
The following are equivalent for a Banach algebra $\mathbb{A}$ with a BAI:

1. $\mathbb{A}$ is pseudo-amenable;
2. $\mathbb{A}$ is approximately amenable.
GNoA’s for group algebras

Theorem (Ghahramani–Loy, 2004; Ghahramani–Zhang, 2007)

The following are equivalent for a locally compact group G:

1. $L^1(G)$ is pseudo-amenable;
2. $L^1(G)$ is approximately amenable;
3. $L^1(G)$ is amenable;
4. G is amenable.

Idea of 1. $\Rightarrow$ 3. for discrete G.

Let $(d_\alpha)_{\alpha} \subset \ell_1(G) \otimes \ell_1(G) \sim = \ell_1(G \times G)$ be an approximate diagonal for $\ell_1(G)$.

Then $(|d_\alpha|, \|d_\alpha\|)_{\alpha} \subset \ell_1(G \times G)$ is a bounded approximate diagonal for $\ell_1(G)$. 
Theorem (Ghahramani–Loy, 2004; Ghahramani–Zhang, 2007)

The following are equivalent for a locally compact group $G$:

1. $L_1(G)$ is pseudo-amenable;
2. $L_1(G)$ is approximately amenable;
3. $L_1(G)$ is amenable;
4. $G$ is amenable.

Idea of 1. $\Rightarrow$ 3. for discrete $G$. Let $(d_\alpha)_{\alpha} \subset \ell_1(G) \otimes \ell_1(G) \sim = \ell_1(G \times G)$ be an approximate diagonal for $\ell_1(G)$. Then $|d_\alpha| \parallel d_\alpha \subset \ell_1(G \times G)$ is a bounded approximate diagonal for $\ell_1(G)$.
Theorem (Ghahramani–Loy, 2004; Ghahramani–Zhang, 2007)

The following are equivalent for a locally compact group $G$:

1. $L_1(G)$ is pseudo-amenable;
2. $L_1(G)$ is approximately amenable;
3. $L_1(G)$ is amenable;
4. $G$ is amenable.

Idea of 1. $\Rightarrow$ 3. for discrete $G$.

Let $(d_\alpha)_{\alpha} \subset L_1(G) \otimes L_1(G) \sim = L_1(G \times G)$ be an approximate diagonal for $L_1(G)$. Then $(|d_\alpha| \parallel d_\alpha)_{\alpha} \subset L_1(G \times G)$ is a bounded approximate diagonal for $L_1(G)$. 
The following are equivalent for a locally compact group $G$:

1. $L^1(G)$ is pseudo-amenable;
The following are equivalent for a locally compact group $G$:

1. $L^1(G)$ is pseudo-amenable;
2. $L^1(G)$ is approximately amenable;
GNoA’s for group algebras

Theorem (Ghahramani–Loy, 2004; Ghahramani–Zhang, 2007)

The following are equivalent for a locally compact group $G$:

1. $L^1(G)$ is pseudo-amenable;
2. $L^1(G)$ is approximately amenable;
3. $L^1(G)$ is amenable;
Theorem (Ghahramani–Loy, 2004; Ghahramani–Zhang, 2007)

The following are equivalent for a locally compact group $G$:

1. $L^1(G)$ is pseudo-amenable;
2. $L^1(G)$ is approximately amenable;
3. $L^1(G)$ is amenable;
4. $G$ is amenable.
Theorem (Ghahramani–Loy, 2004; Ghahramani–Zhang, 2007)

The following are equivalent for a locally compact group $G$:

1. $L^1(G)$ is pseudo-amenable;
2. $L^1(G)$ is approximately amenable;
3. $L^1(G)$ is amenable;
4. $G$ is amenable.

Idea of 1. $\implies$ 3. for discrete $G$. 

Theorem (Ghahramani–Loy, 2004; Ghahramani–Zhang, 2007)

The following are equivalent for a locally compact group $G$:

1. $L^1(G)$ is pseudo-amenable;
2. $L^1(G)$ is approximately amenable;
3. $L^1(G)$ is amenable;
4. $G$ is amenable.

Idea of 1. $\implies$ 3. for discrete $G$.

Let $(d_\alpha)_\alpha \subset \ell^1(G) \otimes \ell^1(G) \cong \ell^1(G \times G)$ be an approximate diagonal for $\ell^1(G)$.
The following are equivalent for a locally compact group $G$:

1. $L^1(G)$ is pseudo-amenable;
2. $L^1(G)$ is approximately amenable;
3. $L^1(G)$ is amenable;
4. $G$ is amenable.

Idea of 1. $\implies$ 3. for discrete $G$.

Let $(d_\alpha)_\alpha \subset \ell^1(G) \otimes \ell^1(G) \cong \ell^1(G \times G)$ be an approximate diagonal for $\ell^1(G)$. Then \( \left( \frac{|d_\alpha|}{\|d_\alpha\|} \right)_\alpha \subset \ell^1(G \times G) \).
Theorem (Ghahramani–Loy, 2004; Ghahramani–Zhang, 2007)

The following are equivalent for a locally compact group $G$:

1. $L^1(G)$ is pseudo-amenable;
2. $L^1(G)$ is approximately amenable;
3. $L^1(G)$ is amenable;
4. $G$ is amenable.

Idea of $1. \implies 3.$ for discrete $G$.

Let $(d_\alpha)_\alpha \subset \ell^1(G) \otimes \ell^1(G) \cong \ell^1(G \times G)$ be an approximate diagonal for $\ell^1(G)$. Then $\left( \frac{|d_\alpha|}{\|d_\alpha\|} \right)_\alpha \subset \ell^1(G \times G)$ is a bounded approximate diagonal for $\ell^1(G)$. 

$\square$
Theorem (Connes, Haagerup, et al.; mid 1970s–mid 1980s)

The following are equivalent for a C*-algebra \( A \):

1. \( A \) is amenable;
2. \( A \) is nuclear;
3. \( A^{**} \) is Connes-amenable, injective, semidiscrete, has Schwartz' Property (P), is approximately finite-dimensional, . . .

Question

Does pseudo-amenability/approximate amenability of a C*-algebra entail its amenability?
GNoA’s for $C^*$-algebras, I

Theorem (Connes, Haagerup, et. al.; mid 1970s–mid 1980s)

The following are equivalent for a $C^*$-algebra $A$:

1. $A$ is amenable;
2. $A$ is nuclear;
3. $A^{**}$ is Connes-amenable, injective, semidiscrete, has Schwartz' Property ($P$), is approximately finite-dimensional, ...
Theorem (Connes, Haagerup, et. al.; mid 1970s–mid 1980s)

The following are equivalent for a $C^*$-algebra $A$:

1. $A$ is amenable;
2. $A$ is nuclear;
3. $A^{**}$ is Connes-amenable, injective, semidiscrete, has Schwartz' Property ($P$), is approximately finite-dimensional, . . .

Question

Does pseudo-amenability/approximate amenability of a $C^*$-algebra entail its amenability?
Theorem (Connes, Haagerup, et. al.; mid 1970s–mid 1980s)

The following are equivalent for a $C^*$-algebra $\mathcal{A}$:

1. $\mathcal{A}$ is amenable;
The following are equivalent for a $C^*$-algebra $\mathcal{A}$:

1. $\mathcal{A}$ is amenable;
2. $\mathcal{A}$ is nuclear;
Theorem (Connes, Haagerup, et. al.; mid 1970s–mid 1980s)

The following are equivalent for a $C^*$-algebra $\mathfrak{A}$:

1. $\mathfrak{A}$ is amenable;
2. $\mathfrak{A}$ is nuclear;
3. $\mathfrak{A}^{**}$ is Connes-amenable,
Theorem (Connes, Haagerup, et. al.; mid 1970s–mid 1980s)

The following are equivalent for a $C^*$-algebra $\mathcal{A}$:

1. $\mathcal{A}$ is amenable;
2. $\mathcal{A}$ is nuclear;
3. $\mathcal{A}^{**}$ is Connes-amenable, injective,
The following are equivalent for a $C^*$-algebra $A$:

1. $A$ is amenable;
2. $A$ is nuclear;
3. $A^{**}$ is Connes-amenable, injective, semidiscrete,
Theorem (Connes, Haagerup, et. al.; mid 1970s–mid 1980s)

The following are equivalent for a $C^*$-algebra $\mathcal{A}$:

1. $\mathcal{A}$ is amenable;
2. $\mathcal{A}$ is nuclear;
3. $\mathcal{A}^{**}$ is Connes-amenable, injective, semidiscrete, has Schwartz’ Property (P),
The following are equivalent for a $C^*$-algebra $\mathcal{A}$:

1. $\mathcal{A}$ is amenable;
2. $\mathcal{A}$ is nuclear;
3. $\mathcal{A}^{**}$ is Connes-amenable, injective, semidiscrete, has Schwartz’ Property ($P$), is approximately finite-dimensional,
Theorem (Connes, Haagerup, et. al.; mid 1970s–mid 1980s)

The following are equivalent for a $C^*$-algebra $\mathfrak{A}$:

1. $\mathfrak{A}$ is amenable;
2. $\mathfrak{A}$ is nuclear;
3. $\mathfrak{A}^{**}$ is Connes-amenable, injective, semidiscrete, has Schwartz’ Property (P), is approximately finite-dimensional, . . .
Theorem (Connes, Haagerup, et. al.; mid 1970s–mid 1980s)

The following are equivalent for a $C^*$-algebra $\mathcal{A}$:

1. $\mathcal{A}$ is amenable;
2. $\mathcal{A}$ is nuclear;
3. $\mathcal{A}^{**}$ is Connes-amenable, injective, semidiscrete, has Schwartz’ Property (P), is approximately finite-dimensional, ...
Theorem (Connes, Haagerup, et. al.; mid 1970s–mid 1980s)

The following are equivalent for a $C^*$-algebra $\mathfrak{A}$:

1. $\mathfrak{A}$ is amenable;
2. $\mathfrak{A}$ is nuclear;
3. $\mathfrak{A}^{**}$ is Connes-amenable, injective, semidiscrete, has Schwartz’ Property (P), is approximately finite-dimensional, . . .

Question

Does pseudo-amenability/approximate amenability of a $C^*$-algebra
GNoA’s for $C^*$-algebras, I

Theorem (Connes, Haagerup, et. al.; mid 1970s–mid 1980s)

The following are equivalent for a $C^*$-algebra $\mathcal{A}$:

1. $\mathcal{A}$ is amenable;
2. $\mathcal{A}$ is nuclear;
3. $\mathcal{A}^{**}$ is Connes-amenable, injective, semidiscrete, has Schwartz’ Property (P), is approximately finite-dimensional, . . .

Question

Does pseudo-amenity/approximate amenability of a $C^*$-algebra entail its amenability?
GNoA’s for $C^*$-algebras, II

Theorem (Ozawa, 2004)

The von Neumann algebras $B(\ell^2)$ and $\ell^\infty \bigoplus_{n=1}^\infty B(\ell^2_n)$ are not pseudo-amenable.

Observation

Let $id_H \in A \subset B(H)$ be a pseudo-amenable $C^*$-algebra. Then there is a net $(T_\alpha)_{\alpha \in E}$ such that $T_\alpha(id_H) = id_H$ for all $\alpha$ and $u \cdot T_\alpha \cdot u^* - T_\alpha \to 0$ ($u \in U(A)$) ($U(A)$ = unitaries in $A$).
Theorem (Ozawa, 2004)

The von Neumann algebras $B(\ell_2)$ and $\ell_\infty\bigoplus_{n=1}^\infty B(\ell_2^n)$ are not pseudo-amenable.

Observation

Let $\text{id}_H \in A \subset B(H)$ be a pseudo-amenable $C^*$-algebra. Then there is a net $\left( T_\alpha \right)_{\alpha \in E}$ such that $T_\alpha(\text{id}_H) = \text{id}_H$ for all $\alpha$ and $u \cdot T_\alpha \cdot u^* - T_\alpha \to 0$ ($u \in U(A)$) ($U(A) = \text{unitaries in } A$).
Theorem (Ozawa, 2004)

*The von Neumann algebras*
Theorem (Ozawa, 2004)

The von Neumann algebras

\[ B(\ell^2) \quad \text{and} \quad \ell^\infty \bigoplus_{n=1}^{\infty} B(\ell^2_n) \]
The von Neumann algebras

\[ \mathcal{B}(\ell^2) \quad \text{and} \quad \ell^\infty \bigoplus_{n=1}^{\infty} \mathcal{B}(\ell^2_n) \]

are not pseudo-amenable.
Theorem (Ozawa, 2004)

The von Neumann algebras

\[ \mathcal{B}(\ell^2) \quad \text{and} \quad \ell^\infty - \bigoplus_{n=1}^{\infty} \mathcal{B}(\ell^2_n) \]

are not pseudo-amenable.

Observation
GNoA’s for $C^*$-algebras, II

Theorem (Ozawa, 2004)

The von Neumann algebras

$$\mathcal{B}(\ell^2) \quad \text{and} \quad \ell^\infty - \bigoplus_{n=1}^\infty \mathcal{B}(\ell^2_n)$$

are not pseudo-amenable.

Observation

Let $\text{id}_H \in \mathcal{A} \subset \mathcal{B}(\mathcal{H})$ be a pseudo-amenable $C^*$-algebra.
The von Neumann algebras

\[ \mathcal{B}(\ell^2) \quad \text{and} \quad \ell^\infty - \bigoplus_{n=1}^{\infty} \mathcal{B}(\ell^2_n) \]

are not pseudo-amenable.

Observation

Let \( \text{id}_\mathcal{H} \in \mathcal{A} \subset \mathcal{B}(\mathcal{H}) \) be a pseudo-amenable \( C^* \)-algebra. Then there is a net \( (T_\alpha)_{\alpha} \subset \mathcal{E}_\mathcal{A}(\mathcal{B}(\mathcal{H})) \)
GNoA’s for $C^*$-algebras, II

**Theorem (Ozawa, 2004)**

The von Neumann algebras

$$B(ℓ^2) \quad \text{and} \quad ℓ^∞ \bigoplus_{n=1}^∞ B(ℓ^{2}_n)$$

are not pseudo-amenable.

**Observation**

Let $\text{id}_{\mathcal{H}} \in \mathcal{A} \subseteq B(\mathcal{H})$ be a pseudo-amenable $C^*$-algebra. Then there is a net $(T_\alpha)_\alpha \subseteq \mathcal{E}_\mathcal{A}(B(\mathcal{H}))$ such that $T_\alpha(\text{id}_{\mathcal{H}}) = \text{id}_{\mathcal{H}}$ for all $\alpha$. 

Advertisements
The von Neumann algebras

$$\mathcal{B}(\ell^2) \quad \text{and} \quad \ell^\infty \bigoplus_{n=1}^{\infty} \mathcal{B}(\ell^2_n)$$

are not pseudo-amenable.

Observation

Let $\text{id}_{\mathcal{H}} \in \mathcal{A} \subset \mathcal{B}(\mathcal{H})$ be a pseudo-amenable $C^*$-algebra. Then there is a net $(T_\alpha)_{\alpha} \subset \mathcal{E}_{\mathcal{A}}(\mathcal{B}(\mathcal{H}))$ such that $T_\alpha(\text{id}_{\mathcal{H}}) = \text{id}_{\mathcal{H}}$ for all $\alpha$ and

$$u \cdot T_\alpha \cdot u^* - T_\alpha \to 0 \quad (u \in \mathcal{U}(\mathcal{A}))$$
Theorem (Ozawa, 2004)

The von Neumann algebras

\[ \mathcal{B}(\ell^2) \quad \text{and} \quad \ell^\infty - \bigoplus_{n=1}^{\infty} \mathcal{B}(\ell^2_n) \]

are not pseudo-amenable.

Observation

Let \( \text{id}_{\mathcal{H}} \in \mathcal{A} \subset \mathcal{B}(\mathcal{H}) \) be a pseudo-amenable \( C^* \)-algebra. Then there is a net \( (T_\alpha)_\alpha \subset \mathcal{E}_{\mathcal{A}}(\mathcal{B}(\mathcal{H})) \) such that \( T_\alpha(\text{id}_{\mathcal{H}}) = \text{id}_{\mathcal{H}} \) for all \( \alpha \) and

\[ u \cdot T_\alpha \cdot u^* - T_\alpha \to 0 \quad (u \in \mathcal{U}(\mathcal{A})) \]

\( (\mathcal{U}(\mathcal{A}) = \text{unitaries in } \mathcal{A}) \).
Complete boundedness and complete positivity, I
Theorem (Paulsen, 1982)

Let $A$ be a unital $C^{*}$-algebra, and let $T \in \text{CB}(A, B(H))$. Then there are $S_1, S_2 \in \text{CP}(A, B(H))$ with $S_1(e_A) = S_2(e_A) = \|T\|_{cb} \text{id}_H$ such that

$$\begin{bmatrix} S_1 & T \\ T^* & S_2 \end{bmatrix} : M_2(A) \rightarrow M_2(B(H)) \cong B(H \oplus H),$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} S_1(a) & T(b) \\ T^*(c) & S_2(d) \end{bmatrix}$$

is completely positive.
Theorem (Paulsen, 1982)

Let $\mathcal{A}$ be a unital $C^*$-algebra,
Theorem (Paulsen, 1982)

Let $A$ be a unital $C^*$-algebra, and let $T \in CB(A, B(H))$. Then there are $S_1, S_2 \in CP(A, B(H))$ with $S_1(e_A) = S_2(e_A) = \|T\|_{cb}$ such that

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} S_1(a)T(b)T^*(c)S_2(d) \end{pmatrix}$

is completely positive.
Theorem (Paulsen, 1982)

Let $\mathcal{A}$ be a unital C*-algebra, and let $T \in CB(\mathcal{A}, \mathcal{B}(\mathcal{H}))$. Then there are $S_1, S_2 \in CP(\mathcal{A}, \mathcal{B}(\mathcal{H}))$
Theorem (Paulsen, 1982)

Let $\mathcal{A}$ be a unital $C^*$-algebra, and let $T \in CB(\mathcal{A}, \mathcal{B}(\mathcal{H}))$. Then there are $S_1, S_2 \in CP(\mathcal{A}, \mathcal{B}(\mathcal{H}))$ with

$$S_1(e_{\mathcal{A}}) = S_2(e_{\mathcal{A}}) = \|T\|_{cb} id_{\mathcal{H}}$$
Theorem (Paulsen, 1982)

Let $\mathcal{A}$ be a unital $C^*$-algebra, and let $T \in CB(\mathcal{A}, B(\mathcal{H}))$. Then there are $S_1, S_2 \in CP(\mathcal{A}, B(\mathcal{H}))$ with

$$S_1(e_{\mathcal{A}}) = S_2(e_{\mathcal{A}}) = \|T\|_{cb} \text{id}_{\mathcal{H}}$$

such that

$$\begin{bmatrix} S_1 & T \\ T^* & S_2 \end{bmatrix} : M_2(\mathcal{A}) \hookrightarrow M_2(\mathcal{B}(\mathcal{H})) \cong \mathcal{B}(\mathcal{H} \oplus \mathcal{H}),$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} S_1(a) & T(b) \\ T^*(c) & S_2(d) \end{bmatrix}$$
Theorem (Paulsen, 1982)

Let $\mathcal{A}$ be a unital $C^*$-algebra, and let $T \in CB(\mathcal{A}, B(\mathcal{H}))$. Then there are $S_1, S_2 \in CP(\mathcal{A}, B(\mathcal{H}))$ with

$$S_1(e_{\mathcal{A}}) = S_2(e_{\mathcal{A}}) = \|T\|_{cb} \mathbf{id}_{\mathcal{H}}$$

such that

$$\begin{bmatrix} S_1 & T \\ T^* & S_2 \end{bmatrix} : M_2(\mathcal{A}) \mapsto M_2(B(\mathcal{H})) \cong B(\mathcal{H} \oplus \mathcal{H}),$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} S_1(a) & T(b) \\ T^*(c) & S_2(d) \end{bmatrix}$$

is completely positive.
Complete boundedness and complete positivity, II
Remark

\[ \|T\|_{cb} = \inf \left\{ \max \{\|S_1\|_{cb}, \|S_2\|_{cb}\} \right\} \]

where the inf is taken over all \( S_1, S_2 \in \text{CP}(A, B(H)) \) such that

\[ \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix} \pm \begin{bmatrix} 0 & T \\ T^* & 0 \end{bmatrix} \in \text{CP}(A, B(H)) \]

Question: How do we get our hand on \( S_1, S_2 \)?
Remark

Note that

\[ \| T \|_{cb} = \inf \left\{ \max \{ \| S_1 \|_{cb}, \| S_2 \|_{cb} \} \right\} \]
Remark

Note that

$$\| T \|_{cb} = \inf \{ \max \{ \| S_1 \|_{cb}, \| S_2 \|_{cb} \} \}$$

where the inf is taken
Remark

Note that

\[ \| T \|_{cb} = \inf \{ \max \{ \| S_1 \|_{cb}, \| S_2 \|_{cb} \} \} \]

where the inf is taken over all \( S_1, S_2 \in \mathcal{CP}(\mathcal{A}, \mathcal{B}(\mathcal{H})) \)
Remark

Note that

\[ \| T \|_{cb} = \inf \{ \max \{ \| S_1 \|_{cb}, \| S_2 \|_{cb} \} \} \]

where the inf is taken over all \( S_1, S_2 \in CP(\mathcal{A}, B(\mathcal{H})) \) such that

\[
\begin{bmatrix}
S_1 & 0 \\
0 & S_2
\end{bmatrix} \pm \begin{bmatrix}
0 & T \\
T^* & 0
\end{bmatrix} \in CP(\mathcal{A}, B(\mathcal{H}))
\]
Remark

Note that

\[ \| T \|_{cb} = \inf \left\{ \max \{ \| S_1 \|_{cb}, \| S_2 \|_{cb} \} \right\} \]

where the inf is taken over all \( S_1, S_2 \in \mathcal{CP}(\mathcal{A}, \mathcal{B}(\mathcal{H})) \) such that

\[
\begin{bmatrix}
S_1 & 0 \\
0 & S_2
\end{bmatrix}
\pm
\begin{bmatrix}
0 & T \\
T^* & 0
\end{bmatrix}
\in \mathcal{CP}(\mathcal{A}, \mathcal{B}(\mathcal{H}))
\]

Question
Remark

Note that

\[ \| T \|_{cb} = \inf \left\{ \max \{ \| S_1 \|_{cb}, \| S_2 \|_{cb} \} \right\} \]

where the inf is taken over all \( S_1, S_2 \in \mathcal{CP}(\mathcal{A}, \mathcal{B}(\mathcal{H})) \) such that

\[
\begin{bmatrix}
S_1 & 0 \\
0 & S_2
\end{bmatrix} \pm \begin{bmatrix}
0 & T \\
T^* & 0
\end{bmatrix} \in \mathcal{CP}(\mathcal{A}, \mathcal{B}(\mathcal{H}))
\]

Question

How do we get our hand on \( S_1, S_2 \)?
Elementary operators, IV

Example

Let $id_{H} \in A \subset B(H)$ be a $C^*$-algebra, and let $T \in E_{\ell} A(B(H))$ with $T(id_{H}) = 1$. Choose $n \in \mathbb{N}$, $a = (a_1, \ldots, a_n), b = (b_1, \ldots, b_n) \in A^n$ such that $T = M_{a,b}$ and $\|T\|_{cb} = \|a_1 a_1^* + \cdots + a_n a_n^*\| = \|b_n^* b_n\|$. The elementary operator $M_2(B(H)) \to M_2(B(H))$, $\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \mapsto \sum_{j=1}^{n} \begin{pmatrix} a_j 0 \\ 0 \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \begin{pmatrix} a_j^* 0 \\ 0 \end{pmatrix}$ is completely positive.
Elementary operators, IV

Example

Let $id_H \in A \subseteq B(H)$ be a $C^*$-algebra, and let $T \in E_{\ell^1}(B(H))$ with $T(id_H) = 1$. Choose $n \in \mathbb{N}, a = (a_1, \ldots, a_n), b = (b_1, \ldots, b_n) \in A^n$ such that $T = M_{a,b}$ and $\|T\|_{cb} = \|a_1^*a_1 + \cdots + a_n^*a_n\| = \|b_1^*b_1 + \cdots + b_n^*b_n\|$. The elementary operator $M_2(B(H)) \to M_2(B(H))$, 

\[
\begin{bmatrix}
  x_{11} & x_{12} \\
  x_{21} & x_{22}
\end{bmatrix} \mapsto
\sum_{j=1}^n \begin{bmatrix}
  a_j^*0 & 0 \\
  0 & b_j^*j
\end{bmatrix} \begin{bmatrix}
  x_{11} & x_{12} \\
  x_{21} & x_{22}
\end{bmatrix} \begin{bmatrix}
  a_j0 & 0 \\
  0 & b_jj
\end{bmatrix}
\]

is completely positive.
Example

Let $\text{id}_{\mathcal{H}} \in \mathcal{A} \subset B(\mathcal{H})$ be a $C^*$-algebra,
Example

Let $\text{id}_{\mathcal{H}} \in \mathcal{A} \subset \mathcal{B}(\mathcal{H})$ be a $C^*$-algebra, and let $T \in \mathcal{E}_{\mathcal{A}}(\mathcal{B}(\mathcal{H}))$ with $T(\text{id}_{\mathcal{H}}) = 1$.
Example

Let $\text{id}_\mathcal{H} \in \mathcal{A} \subset \mathcal{B}(\mathcal{H})$ be a $C^*$-algebra, and let $T \in \mathcal{E}_{\mathcal{A}}(\mathcal{B}(\mathcal{H}))$ with $T(\text{id}_\mathcal{H}) = 1$. Choose $n \in \mathbb{N}$,
Example

Let $\text{id}_{\mathcal{H}} \in \mathcal{A} \subset \mathcal{B}(\mathcal{H})$ be a $C^*$-algebra, and let $T \in \mathcal{E}_{\mathcal{A}}(\mathcal{B}(\mathcal{H}))$ with $T(\text{id}_{\mathcal{H}}) = 1$. Choose $n \in \mathbb{N}$, $a = (a_1, \ldots, a_n), b = (b_1, \ldots, b_n) \in \mathcal{A}^n$.
Example

Let $\text{id}_{\mathcal{H}} \in \mathcal{A} \subset B(\mathcal{H})$ be a $C^*$-algebra, and let $T \in \mathcal{E}_\mathcal{A}(B(\mathcal{H}))$ with $T(\text{id}_{\mathcal{H}}) = 1$. Choose $n \in \mathbb{N}$, $a = (a_1, \ldots, a_n)$, $b = (b_1, \ldots, b_n) \in \mathcal{A}^n$ such that $T = M_{a,b}$. 
Elementary operators, IV

Example

Let $\text{id}_{\mathcal{H}} \in \mathcal{A} \subset \mathcal{B}(\mathcal{H})$ be a $C^*$-algebra, and let $T \in \mathcal{E}_{\mathcal{A}}(\mathcal{B}(\mathcal{H}))$ with $T(\text{id}_{\mathcal{H}}) = 1$. Choose $n \in \mathbb{N}$, $a = (a_1, \ldots, a_n), b = (b_1, \ldots, b_n) \in \mathcal{A}^n$ such that $T = M_{a,b}$ and

$$
\| T \|_{cb} = \| a_1 a_1^* + \cdots + a_n a_n^* \| = \| b_1^* b_1 + \cdots + b_n^* b_n \|.
$$
Example

Let \( \text{id}_\mathcal{H} \in \mathcal{A} \subset \mathcal{B}(\mathcal{H}) \) be a \( C^* \)-algebra, and let \( T \in \mathcal{E}_\mathcal{A}(\mathcal{B}(\mathcal{H})) \) with \( T(\text{id}_\mathcal{H}) = 1 \). Choose \( n \in \mathbb{N} \), \( a = (a_1, \ldots, a_n) \), \( b = (b_1, \ldots, b_n) \) \( \in \mathcal{A}^n \) such that \( T = M_{a,b} \) and

\[
\| T \|_{cb} = \| a_1 a_1^* + \cdots + a_n a_n^* \| = \| b_1^* b_1 + \cdots + b_n^* b_n \|.
\]

The elementary operator

\[
M_2(\mathcal{B}(\mathcal{H})) \to M_2(\mathcal{B}(\mathcal{H})),
\]

\[
\begin{bmatrix}
x_{11} & x_{12} \\
x_{21} & x_{22}
\end{bmatrix} \mapsto \sum_{j=1}^{n} \begin{bmatrix}
a_j & 0 \\
0 & b_j^*
\end{bmatrix}
\begin{bmatrix}
x_{11} & x_{12} \\
x_{21} & x_{22}
\end{bmatrix}
\begin{bmatrix}
a_j^* & 0 \\
0 & b_j
\end{bmatrix}
\]
Example

Let $\text{id}_{\mathcal{H}} \in \mathcal{A} \subset \mathcal{B}(\mathcal{H})$ be a $C^*$-algebra, and let $T \in \mathcal{E}_{\mathcal{A}}(\mathcal{B}(\mathcal{H}))$ with $T(\text{id}_{\mathcal{H}}) = 1$. Choose $n \in \mathbb{N}$, $a = (a_1, \ldots, a_n), b = (b_1, \ldots, b_n) \in \mathcal{A}^n$ such that $T = M_{a,b}$ and

$$\| T \|_{cb} = \| a_1 a_1^* + \cdots + a_n a_n^* \| = \| b_1^* b_1 + \cdots + b_n^* b_n \|.$$ 

The elementary operator

$$M_2(\mathcal{B}(\mathcal{H})) \to M_2(\mathcal{B}(\mathcal{H})),$$

$$\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \mapsto \sum_{j=1}^{n} \begin{bmatrix} a_j & 0 \\ 0 & b_j^* \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \begin{bmatrix} a_j^* \\ 0 \end{bmatrix}$$

is completely positive.
Elementary operators, V

Example (continued)

\[
\begin{bmatrix}
a_0 & 0 \\
0 & b^* \\
\end{bmatrix}
\begin{bmatrix}
x_{11} \\
x_{12} \\
\end{bmatrix}
\begin{bmatrix}
a^*_0 & 0 \\
0 & b \\
\end{bmatrix} =
\begin{bmatrix}
a x_{11} \\
a^* x_{12} \\
b x_{21} \\
b^* x_{22} \\
\end{bmatrix}.
\]

So \( S_1 = M_{a, a^*} \) and \( S_2 = M_{b^*, b} \) will do.
Example (continued)

Now

\[
\begin{bmatrix}
0 & 0 \\
0 & 1 \\
1 & 0 \\
0 & 0
\end{bmatrix}
\]

will do.
Example (continued)

Now

$$\begin{bmatrix} a & 0 \\ 0 & b^* \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \begin{bmatrix} a^* & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} ax_{11}a^* & ax_{12}b \\ b^*x_{21}a^* & b^*x_{22}b \end{bmatrix}.$$
Elementary operators, V

Example (continued)

Now

\[
\begin{bmatrix}
  a & 0 \\
  0 & b^*
\end{bmatrix}
\begin{bmatrix}
  x_{11} & x_{12} \\
  x_{21} & x_{22}
\end{bmatrix}
\begin{bmatrix}
  a^* & 0 \\
  0 & b
\end{bmatrix}
= \begin{bmatrix}
  ax_{11}a^* & ax_{12}b \\
  b^*x_{21}a^* & b^*x_{22}b
\end{bmatrix}.
\]

So
Now
\[
\begin{bmatrix}
  a & 0 \\
  0 & b^*
\end{bmatrix}
\begin{bmatrix}
  x_{11} & x_{12} \\
  x_{21} & x_{22}
\end{bmatrix}
\begin{bmatrix}
  a^* & 0 \\
  0 & b
\end{bmatrix} = \begin{bmatrix}
  ax_{11}a^* & ax_{12}b \\
  b^*x_{21}a^* & b^*x_{22}b
\end{bmatrix}.
\]

So
\[S_1 = M_{a,a^*} \quad \text{and} \quad S_2 = M_{b^*,b}\]
Example (continued)

Now

\[
\begin{bmatrix}
    a & 0 \\
    0 & b^* \\
\end{bmatrix}
\begin{bmatrix}
    x_{11} & x_{12} \\
    x_{21} & x_{22} \\
\end{bmatrix}
\begin{bmatrix}
    a^* & 0 \\
    0 & b \\
\end{bmatrix}
= 
\begin{bmatrix}
    ax_{11}a^* & ax_{12}b \\
    b^*x_{21}a^* & b^*x_{22}b \\
\end{bmatrix}.
\]

So

\[S_1 = M_{a,a^*} \quad \text{and} \quad S_2 = M_{b^*,b}\]

will do.
Elementary operators, VI
Consequence
Consequence

Let $n \in \mathcal{A}$
Consequence

Let \( n \in \mathcal{A} \) and \( a, b, c, d \in \mathcal{A}^n \) suitable.
Consequence

Let $n \in \mathcal{A}$ and $a, b, c, d \in \mathcal{A}^n$ suitable. Then

$$\|\begin{bmatrix} M^*_{a,a} & 0 \\ 0 & M^*_{b^*,b} \end{bmatrix} - \begin{bmatrix} M^*_{c,c} & 0 \\ 0 & M^*_{d^*,d} \end{bmatrix}\|_{cb} \leq \|M_{a,b} - M_{c,d}\|_{cb}$$
Consequence

Let \( n \in \mathcal{A} \) and \( a, b, c, \varphi \in \mathcal{A}^n \) suitable. Then

\[
\left\| \begin{bmatrix} M_{a,a}^* & 0 \\ 0 & M_{b^*,b}^* \end{bmatrix} - \begin{bmatrix} M_{c,c}^* & 0 \\ 0 & M_{\varphi^*,\varphi}^* \end{bmatrix} \right\|_{cb} \leq \| M_{a,b} - M_{c,\varphi} \|_{cb}
\]

and thus

\[
\left\| \frac{1}{2} (M_{a,a}^* + M_{b^*,b}^*) - \frac{1}{2} (M_{c,c}^* + M_{\varphi^*,\varphi}^*) \right\|_{cb} \leq \| M_{a,b} - M_{c,\varphi} \|_{cb}
\]
Let $id_H \in A \subset B(H)$ be a $C^*$-algebra. For $c = (c_1, \ldots, c_n) \in A$ and $a \in A$, set $a \cdot c := (ac_1, \ldots, ac_n)$ and $c \cdot a := (c_1a, \ldots, c_na)$.

Let $T = M_{a, b} \in E\ell A(B(H))$, and note that $a \cdot T \cdot b = M_{a \cdot a, b \cdot b}(a, b) \in A$.

Suppose that $T(id_H) = id_H$, and that $a, b \in A$ are suitable for $T$. Set $|T| := \frac{1}{2}(M_{a, a^*} + M_{b, b^*})$.

Let $u \in U(A)$. Then $u \cdot a, b \cdot u^*$ are suitable for $u \cdot T \cdot u^*$ with $|u \cdot T \cdot u^*| = u \cdot |T| \cdot u^*$, so that

$$\|u \cdot |T| \cdot u^* - |T|\|_{cb} \leq \|u \cdot T \cdot u^* - T\|_{cb}.$$
Let $\text{id}_\mathcal{H} \in \mathcal{A} \subset \mathcal{B}(\mathcal{H})$ be a $C^*$-algebra.
Let $\text{id}_{\mathcal{H}} \in \mathcal{A} \subset B(\mathcal{H})$ be a $C^*$-algebra. For $c = (c_1, \ldots, c_n) \in \mathcal{A}$
Let $\text{id}_{\mathcal{H}} \in \mathcal{A} \subset B(\mathcal{H})$ be a $C^*$-algebra. For $c = (c_1, \ldots, c_n) \in \mathcal{A}$ and $a \in \mathcal{A}$,
Let $\text{id}_\mathcal{H} \in \mathcal{A} \subset \mathcal{B}(\mathcal{H})$ be a $C^*$-algebra. For $c = (c_1, \ldots, c_n) \in \mathcal{A}$ and $a \in \mathcal{A}$, set

$$a \cdot c := (ac_1, \ldots, ac_n) \quad \text{and} \quad c \cdot a := (c_1a, \ldots, c_na).$$
Let $\text{id}_H \in \mathcal{A} \subset \mathcal{B}(\mathcal{H})$ be a $C^*$-algebra. For $c = (c_1, \ldots, c_n) \in \mathcal{A}$ and $a \in \mathcal{A}$, set
\[ a \cdot c := (ac_1, \ldots, ac_n) \quad \text{and} \quad c \cdot a := (c_1a, \ldots, c_na). \]
Let $T = M_{a,b} \in \mathcal{E}_\mathcal{A}(\mathcal{B}(\mathcal{H}))$,
Let \( \text{id}_{\mathcal{H}} \in \mathcal{A} \subset \mathcal{B}(\mathcal{H}) \) be a C*-algebra. For \( c = (c_1, \ldots, c_n) \in \mathcal{A} \) and \( a \in \mathcal{A} \), set
\[
a \cdot c := (ac_1, \ldots, ac_n) \quad \text{and} \quad c \cdot a := (c_1a, \ldots, c_na).
\]
Let \( T = M_{a,b} \in \mathcal{E}_{\mathcal{A}}(\mathcal{B}(\mathcal{H})) \), and note that
Let $\text{id}_\mathcal{H} \in \mathcal{A} \subset \mathcal{B}(\mathcal{H})$ be a $C^*$-algebra. For $c = (c_1, \ldots, c_n) \in \mathcal{A}$ and $a \in \mathcal{A}$, set

$$a \cdot c := (ac_1, \ldots, ac_n) \quad \text{and} \quad c \cdot a := (c_1a, \ldots, c_na).$$

Let $T = M_{a,b} \in \mathcal{E}_\mathcal{A}(\mathcal{B}(\mathcal{H}))$, and note that

$$a \cdot T \cdot b = M_{a \cdot a, b \cdot b} \quad (a, b \in \mathcal{A}).$$
Let \( \text{id}_\mathcal{H} \in \mathcal{A} \subset \mathcal{B}(\mathcal{H}) \) be a \( C^* \)-algebra. For \( c = (c_1, \ldots, c_n) \in \mathcal{A} \) and \( a \in \mathcal{A} \), set
\[
a \cdot c := (ac_1, \ldots, ac_n) \quad \text{and} \quad c \cdot a := (c_1a, \ldots, c_na).
\]

Let \( T = M_{a,b} \in \mathcal{E}_\mathcal{A}(\mathcal{B}(\mathcal{H})) \), and note that
\[
a \cdot T \cdot b = M_{a \cdot a, b \cdot b} \quad (a, b \in \mathcal{A}).
\]

Suppose that \( T(\text{id}_\mathcal{H}) = \text{id}_\mathcal{H} \).
GNoA's for C*-algebras, III

Let \( \text{id}_{\mathcal{H}} \in \mathcal{B} \subset \mathcal{B} (\mathcal{H}) \) be a C*-algebra. For \( \mathbf{c} = (c_1, \ldots, c_n) \in \mathcal{A} \) and \( a \in \mathcal{A} \), set

\[
a \cdot \mathbf{c} := (ac_1, \ldots, ac_n) \quad \text{and} \quad \mathbf{c} \cdot a := (c_1a, \ldots, c_na).
\]

Let \( T = M_{a,b} \in \mathcal{E}_{\mathcal{A}}(\mathcal{B}(\mathcal{H})) \), and note that

\[
a \cdot T \cdot b = M_{a \cdot a,b \cdot b} \quad (a, b \in \mathcal{A}).
\]

Suppose that \( T(\text{id}_{\mathcal{H}}) = \text{id}_{\mathcal{H}} \), and that \( a, b \in \mathcal{A}^n \) are suitable for \( T \).
Let \( \text{id}_{\mathcal{H}} \in \mathcal{A} \subseteq \mathcal{B}(\mathcal{H}) \) be a \( C^* \)-algebra. For \( c = (c_1, \ldots, c_n) \in \mathcal{A} \) and \( a \in \mathcal{A} \), set
\[
a \cdot c := (ac_1, \ldots, ac_n) \quad \text{and} \quad c \cdot a := (c_1a, \ldots, c_na).
\]

Let \( T = M_{a,b} \in \mathcal{E}_{\mathcal{A}}(\mathcal{B}(\mathcal{H})) \), and note that
\[
a \cdot T \cdot b = M_{a\cdot a, b\cdot b} \quad (a, b \in \mathcal{A}).
\]

Supppose that \( T(\text{id}_{\mathcal{H}}) = \text{id}_{\mathcal{H}} \), and that \( a, b \in \mathcal{A}^n \) are suitable for \( T \). Set
\[
|T| := \frac{1}{2}(M_{a,a^*} + M_{b^*,b^*}).
\]
Let $\text{id}_{\mathcal{H}} \in \mathcal{A} \subset \mathcal{B}(\mathcal{H})$ be a $C^*$-algebra. For $c = (c_1, \ldots, c_n) \in \mathcal{A}$ and $a \in \mathcal{A}$, set

$$a \cdot c := (ac_1, \ldots, ac_n) \quad \text{and} \quad c \cdot a := (c_1a, \ldots, c_n a).$$

Let $T = M_{a,b} \in \mathcal{E}_\mathcal{A}(\mathcal{B}(\mathcal{H}))$, and note that

$$a \cdot T \cdot b = M_{a \cdot a, b \cdot b} \quad (a, b \in \mathcal{A}).$$

Supppose that $T(\text{id}_{\mathcal{H}}) = \text{id}_{\mathcal{H}}$, and that $a, b \in \mathcal{A}^n$ are suitable for $T$. Set

$$|T| := \frac{1}{2}(M_{a,a^*} + M_{b^*,b}).$$

Let $u \in \mathcal{U}(\mathcal{A})$. 

\[ \text{GNoA's for } C^*-\text{algebras, III} \]
Let $\text{id}_{\mathcal{H}} \in \mathcal{A} \subset \mathcal{B}(\mathcal{H})$ be a $C^*$-algebra. For $c = (c_1, \ldots, c_n) \in \mathcal{A}$ and $a \in \mathcal{A}$, set

$$a \cdot c := (ac_1, \ldots, ac_n) \quad \text{and} \quad c \cdot a := (c_1 a, \ldots, c_n a).$$

Let $T = M_{a,b} \in \mathcal{E}_{\mathcal{A}}(\mathcal{B}(\mathcal{H}))$, and note that

$$a \cdot T \cdot b = M_{a \cdot a, b \cdot b} \quad (a, b \in \mathcal{A}).$$

Suppose that $T(\text{id}_{\mathcal{H}}) = \text{id}_{\mathcal{H}}$, and that $a, b \in \mathcal{A}^n$ are suitable for $T$. Set

$$|T| := \frac{1}{2}(M_{a,a^*} + M_{b^*,b}).$$

Let $u \in \mathcal{U}(\mathcal{A})$. Then $u \cdot a, b \cdot u^*$
Let $\text{id}_{\mathcal{H}} \in \mathcal{A} \subset \mathcal{B}(\mathcal{H})$ be a $C^*$-algebra. For $c = (c_1, \ldots, c_n) \in \mathcal{A}$ and $a \in \mathcal{A}$, set

$$a \cdot c := (ac_1, \ldots, ac_n) \quad \text{and} \quad c \cdot a := (c_1 a, \ldots, c_n a).$$

Let $T = M_{a,b} \in \mathcal{E}\ell_{\mathcal{A}}(\mathcal{B}(\mathcal{H}))$, and note that

$$a \cdot T \cdot b = M_{a \cdot a, b \cdot b} \quad (a, b \in \mathcal{A}).$$

Suppose that $T(\text{id}_{\mathcal{H}}) = \text{id}_{\mathcal{H}}$, and that $a, b \in \mathcal{A}^n$ are suitable for $T$. Set

$$|T| := \frac{1}{2}(M_{a,a^*} + M_{b^*, b}).$$

Let $u \in \mathcal{U}(\mathcal{A})$. Then $u \cdot a, b \cdot u^*$ are suitable for $u \cdot T \cdot u^*$.
Let $\text{id}_{\mathcal{H}} \in \mathcal{A} \subset \mathcal{B}(\mathcal{H})$ be a $C^*$-algebra. For $c = (c_1, \ldots, c_n) \in \mathcal{A}$ and $a \in \mathcal{A}$, set
\[
a \cdot c := (ac_1, \ldots, ac_n) \quad \text{and} \quad c \cdot a := (c_1a, \ldots, c_na).
\]

Let $T = M_{a,b} \in \ell_2(\mathcal{B}(\mathcal{H}))$, and note that
\[
a \cdot T \cdot b = M_{a \cdot a,b \cdot b} \quad (a, b \in \mathcal{A}).
\]

Suppose that $T(\text{id}_{\mathcal{H}}) = \text{id}_{\mathcal{H}}$, and that $a, b \in \mathcal{A}^n$ are suitable for $T$. Set
\[
|T| := \frac{1}{2}(M_{a,a^*} + M_{b^*,b}).
\]

Let $u \in \mathcal{U}(\mathcal{A})$. Then $u \cdot a, b \cdot u^*$ are suitable for $u \cdot T \cdot u^*$ with
\[
|u \cdot T \cdot u^*| = u \cdot |T| \cdot u^*.
\]
Let $\text{id}_{\mathcal{H}} \in \mathcal{A} \subset \mathcal{B}(\mathcal{H})$ be a $C^*$-algebra. For $c = (c_1, \ldots, c_n) \in \mathcal{A}$ and $a \in \mathcal{A}$, set

$$a \cdot c := (ac_1, \ldots, ac_n) \quad \text{and} \quad c \cdot a := (c_1a, \ldots, c_na).$$

Let $T = M_{a,b} \in \mathcal{E}_\mathcal{A}(\mathcal{B}(\mathcal{H}))$, and note that

$$a \cdot T \cdot b = M_{a \cdot a,b \cdot b} \quad (a, b \in \mathcal{A}).$$

Supppose that $T(\text{id}_{\mathcal{H}}) = \text{id}_{\mathcal{H}}$, and that $a, b \in \mathcal{A}^n$ are suitable for $T$. Set

$$|T| := \frac{1}{2}(M_{a,a^*} + M_{b^*,b}).$$

Let $u \in \mathcal{U}(\mathcal{A})$. Then $u \cdot a, b \cdot u^*$ are suitable for $u \cdot T \cdot u^*$ with

$$|u \cdot T \cdot u^*| = u \cdot |T| \cdot u^*,$$

so that

$$\|u \cdot |T| \cdot u^* - |T|\|_{cb} \leq \|u \cdot T \cdot u^* - T\|_{cb}.$$
A von Neumann algebra $M \subset B(H)$ is called injective if there is a norm one projection $C: B(H) \rightarrow M'$. 

Theorem 

Let $id_H \in A \subset B(H)$ be pseudo-amenable $C^*$-algebra. Then $A'$ is injective. 

Proof. 

Let $(T_\alpha)_{\alpha \in A} \in A \subset \ell^A(B(H))$ such that $T_\alpha(id_H) = id_H$ ($\alpha \in A$) and $u \cdot T_\alpha \cdot u^* - T_\alpha \rightarrow 0$ ($u \in U(A)$).
A von Neumann algebra $M \subset B(H)$ is called injective if there is a norm one projection $C : B(H) \to M'$. 

Theorem

Let $id_H \in A \subset B(H)$ be pseudo-amenable $C^*$-algebra. Then $A'$ is injective.

Proof.

Let $(T_\alpha)_{\alpha \in A} \in A \subset E_{\ell A}(B(H))$ such that $T_\alpha(id_H) = id_H$ and $u \cdot T_\alpha(u)^* - T_\alpha \to 0$ ($u \in U(A)$).
GNoA’s for $C^*$-algebras, IV

Definition

A von Neumann algebra $M \subset B(H)$
A von Neumann algebra $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ is called injective.
Definition

A von Neumann algebra $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ is called \textit{injective} if there is a norm one projection $\mathcal{C} : \mathcal{B}(\mathcal{H}) \to \mathcal{M}'$. 
GNoA’s for $C^*$-algebras, IV

**Definition**
A von Neumann algebra $M \subset B(H)$ is called **injective** if there is a norm one projection $C : B(H) \to M'$.

**Theorem**

Let $id_H \in A \subset B(H)$ be pseudo-amenable $C^*$-algebra. Then $A'$ is injective.

**Proof.** Let $(T_\alpha)_{\alpha \in A} \in A \subset E\ell A(B(H))$ such that $T_\alpha(id_H) = id_H$ $(\alpha \in A)$ and $u \cdot T_\alpha \cdot u^* - T_\alpha \to 0$ $(u \in U(A))$. 

Advertisements
Definition
A von Neumann algebra $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ is called **injective** if there is a norm one projection $\mathcal{C} : \mathcal{B}(\mathcal{H}) \to \mathcal{M}'$.

Theorem
*Let $\text{id}_\mathcal{H} \in \mathcal{A} \subset \mathcal{B}(\mathcal{H})$ be pseudo-amenable $C^*$-algebra.*
GNoA’s for $C^*$-algebras, IV

**Definition**

A von Neumann algebra $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ is called **injective** if there is a norm one projection $C : \mathcal{B}(\mathcal{H}) \to \mathcal{M}'$.

**Theorem**

Let $\text{id}_\mathcal{H} \in \mathcal{A} \subset \mathcal{B}(\mathcal{H})$ be pseudo-amenable $C^*$-algebra. Then $\mathcal{A}'$ is injective.
Definition

A von Neumann algebra $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ is called injective if there is a norm one projection $C : \mathcal{B}(\mathcal{H}) \to \mathcal{M}'$.

Theorem

Let $\text{id}_{\mathcal{H}} \in \mathcal{A} \subset \mathcal{B}(\mathcal{H})$ be pseudo-amenable $C^*$-algebra. Then $\mathcal{A}'$ is injective.

Proof.
A von Neumann algebra $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ is called **injective** if there is a norm one projection $C : \mathcal{B}(\mathcal{H}) \to \mathcal{M}'$.

Let $\text{id}_{\mathcal{H}} \in \mathcal{A} \subset \mathcal{B}(\mathcal{H})$ be pseudo-amenable $C^*$-algebra. Then $\mathcal{A}'$ is injective.

Let $(T_\alpha)_{\alpha \in \mathcal{A}} \subset \mathcal{E}_\mathcal{A}(\mathcal{B}(\mathcal{H}))$
Definition

A von Neumann algebra $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ is called \textit{injective} if there is a norm one projection $C : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{M}'$.

Theorem

Let $\text{id}_\mathcal{H} \in \mathcal{A} \subset \mathcal{B}(\mathcal{H})$ be pseudo-amenable $C^*$-algebra. Then $\mathcal{A}'$ is injective.

Proof.

Let $(T_\alpha)_{\alpha \in \mathcal{A}} \subset \mathcal{E}l_{\mathcal{A}}(\mathcal{B}(\mathcal{H}))$ such that
Definition

A von Neumann algebra $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ is called injective if there is a norm one projection $C : \mathcal{B}(\mathcal{H}) \to \mathcal{M}'$.

Theorem

Let $\text{id}_{\mathcal{H}} \in \mathcal{A} \subset \mathcal{B}(\mathcal{H})$ be pseudo-amenable $C^*$-algebra. Then $\mathcal{A}'$ is injective.

Proof.

Let $(T_\alpha)_{\alpha \in \mathcal{A}} \subset \mathcal{E}_\mathcal{A}(\mathcal{B}(\mathcal{H}))$ such that

$$T_\alpha(\text{id}_{\mathcal{H}}) = \text{id}_{\mathcal{H}} \quad (\alpha \in \mathcal{A})$$
Definition

A von Neumann algebra $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ is called **injective** if there is a norm one projection $C : \mathcal{B}(\mathcal{H}) \to \mathcal{M}'$.

Theorem

Let $\text{id}_\mathcal{H} \in \mathcal{A} \subset \mathcal{B}(\mathcal{H})$ be pseudo-amenable $C^*$-algebra. Then $\mathcal{A}'$ is injective.

Proof.

Let $(T_\alpha)_{\alpha \in \mathcal{A}} \subset \mathcal{E}_{\mathcal{A}}(\mathcal{B}(\mathcal{H}))$ such that

$$T_\alpha(\text{id}_\mathcal{H}) = \text{id}_\mathcal{H} \quad (\alpha \in \mathcal{A})$$

and

$$u \cdot T_\alpha \cdot u^* - T_\alpha \to 0 \quad (u \in \mathcal{U}(\mathcal{A})).$$
Choose a corresponding net \((|T_\alpha|)\alpha\in A \subset E \ell A(B(H))\) of Wittstock moduli such that \(|T_\alpha| (\text{id}_H) = \|T_\alpha\|_{cb} \text{id}_H(\alpha \in A)\) and \(u \cdot |T_\alpha| \cdot u^* - |T_\alpha| \to 0\) \((u \in \mathbb{U}(A))\).

Let \(U\) be an ultrafilter over \(A\) that dominates the order filter. Define \(E : B(H) \to B(H), x \mapsto \lim\limits_{\alpha \to U} \|T_\alpha\| |T_\alpha| (x)\).

The \(\|E\| = 1\), \(E|A = \text{id}_A\)', and \(E(B(H)) \subset A'\).
Proof (continued).

Choose a corresponding net \((|T_\alpha|)\alpha \in A \subset E_\ell(A)(B(H))\) of Wittstock moduli such that \(|T_\alpha| (id_H) = \|T_\alpha\|_{cb} id_H\) and 

\[u \cdot |T_\alpha| \cdot u^* \to 0 \quad (u \in U(A)).\]

Let \(U\) be an ultrafilter over \(A\) that dominates the order filter. Define \(E : B(H) \to B(H), x \mapsto \limsup_{\alpha \to U} 1_{|T_\alpha|} \|T_\alpha\| x\). The \(\|E\| = 1\), \(E|A' = id_{A'}\), and \(E(B(H)) \subset A'\).
Proof (continued).

Choose a corresponding net \((|T_\alpha|)_{\alpha \in A} \subset \text{Ell}_\mathbb{A}(B(H))\) of Wittstock moduli
Proof (continued).

Choose a corresponding net \((|T_\alpha|)_{\alpha \in A} \subset \mathcal{E}_\mathfrak{A}(\mathcal{B}(\mathcal{H}))\) of Wittstock moduli such that
Proof (continued).

Choose a corresponding net \((|T_\alpha|)_{\alpha \in \mathbb{A}} \subset \mathcal{E}_{\mathbb{A}}(\mathcal{B}(\mathcal{H}))\) of Wittstock moduli such that

\[
|T_\alpha|(id_{\mathcal{H}}) = \|T_\alpha\|_{cb} \text{id}_{\mathcal{H}} \quad (\alpha \in \mathbb{A})
\]
Proof (continued).

Choose a corresponding net \(|T_\alpha|\)\(\alpha \in \mathbb{A}\) \(\subset \mathcal{EL}_\mathbb{A}(\mathcal{B}(\mathcal{H}))\) of Wittstock moduli such that

\[ |T_\alpha|(\text{id}_\mathcal{H}) = \|T_\alpha\|_{\text{cb}} \text{id}_\mathcal{H} \quad (\alpha \in \mathbb{A}) \]

and

\[ u \cdot |T_\alpha| \cdot u^* - |T_\alpha| \rightarrow 0 \quad (u \in \mathcal{U}(\mathbb{A})). \]
Choose a corresponding net \( (| T_\alpha |)_{\alpha \in A} \subset \mathcal{E}_{\mathcal{A}}(\mathcal{B}(\mathcal{H})) \) of Wittstock moduli such that
\[
| T_\alpha |(\text{id}_{\mathcal{H}}) = \| T_\alpha \|_{\text{cb}} \text{id}_{\mathcal{H}} \quad (\alpha \in A)
\]
and
\[
u \cdot | T_\alpha | \cdot u^* - | T_\alpha | \to 0 \quad (u \in \mathcal{U}(\mathcal{A})).
\]
Let \( \mathcal{U} \) be an ultrafilter over \( A \).
Proof (continued).

Choose a corresponding net \((|T_\alpha|)_{\alpha \in \mathbb{A}} \subset \mathcal{E}_{\mathbb{A}}(\mathcal{B}(\mathcal{H}))\) of Wittstock moduli such that

\[
|T_\alpha|(id_{\mathcal{H}}) = \|T_\alpha\|_{cb} id_{\mathcal{H}} \quad (\alpha \in \mathbb{A})
\]

and

\[
u \cdot |T_\alpha| \cdot u^* - |T_\alpha| \to 0 \quad (u \in \mathcal{U}(\mathcal{A})).
\]

Let \(\mathcal{U}\) be an ultrafilter over \(\mathbb{A}\) that dominates the order filter.
Proof (continued).

Choose a corresponding net \(|T_\alpha|\)_{\alpha \in \mathbb{A}} \subseteq \mathcal{E}_\mathbb{A}(\mathbb{B}(\mathcal{H})) of Wittstock moduli such that

\[ |T_\alpha|(\text{id}_\mathcal{H}) = \|T_\alpha\|_{cb} \text{id}_\mathcal{H} \quad (\alpha \in \mathbb{A}) \]

and

\[ u \cdot |T_\alpha| \cdot u^* - |T_\alpha| \to 0 \quad (u \in \mathcal{U}(\mathbb{A})). \]

Let \( \mathcal{U} \) be an ultrafilter over \( \mathbb{A} \) that dominates the order filter. Define

\[ \mathcal{E} : \mathbb{B}(\mathcal{H}) \to \mathbb{B}(\mathcal{H}), \quad x \mapsto \text{WOT-} \lim_{\alpha \to \mathcal{U}} \frac{1}{\|T_\alpha\|} |T_\alpha|(x). \]
Proof (continued).

Choose a corresponding net \((|T_\alpha|)_{\alpha \in A} \subset \mathcal{E}_{\mathcal{A}}(\mathcal{B}(\mathcal{H}))\) of Wittstock moduli such that

\[
|T_\alpha|(\text{id}_{\mathcal{H}}) = \|T_\alpha\|_{\text{cb}} \text{id}_{\mathcal{H}} \quad (\alpha \in A)
\]

and

\[
u \cdot |T_\alpha| \cdot u^* - |T_\alpha| \to 0 \quad (u \in \mathcal{U}(\mathcal{A})).
\]

Let \(\mathcal{U}\) be an ultrafilter over \(A\) that dominates the order filter. Define

\[
\mathcal{E} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}), \quad x \mapsto \text{WOT- lim}_{\alpha \to \mathcal{U}} \frac{1}{\|T_\alpha\|} |T_\alpha|(x).
\]

The \(\|\mathcal{E}\| = 1\),
Proof (continued).

Choose a corresponding net \((|T_\alpha|)_{\alpha \in A} \subset \mathcal{E}_\mathcal{A}(B(\mathcal{H}))\) of Wittstock moduli such that

\[
|T_\alpha|(id_{\mathcal{H}}) = \|T_\alpha\|_{cb} id_{\mathcal{H}} \quad (\alpha \in A)
\]

and

\[
u \cdot |T_\alpha| \cdot u^* - |T_\alpha| \to 0 \quad (u \in \mathcal{U}(A)).
\]

Let \(\mathcal{U}\) be an ultrafilter over \(A\) that dominates the order filter. Define

\[
\mathcal{E} : B(\mathcal{H}) \to B(\mathcal{H}), \quad x \mapsto \text{WOT- lim}_{\alpha \to \mathcal{U}} \frac{1}{\|T_\alpha\|} |T_\alpha|(x).
\]

The \(\|\mathcal{E}\| = 1, \mathcal{E}|_{\mathcal{U}'} = \text{id}_{\mathcal{U}'}\).
Proof (continued).

Choose a corresponding net \((|T_\alpha|)_{\alpha \in \mathbb{A}} \subset \mathcal{E}_\mathbb{A}(\mathcal{B}(\mathcal{H}))\) of Wittstock moduli such that

\[ |T_\alpha|(id_\mathcal{H}) = \|T_\alpha\|_{cb} id_\mathcal{H} \quad (\alpha \in \mathbb{A}) \]

and

\[ u \cdot |T_\alpha| \cdot u^* - |T_\alpha| \to 0 \quad (u \in \mathcal{U}(\mathbb{A})). \]

Let \(\mathcal{U}\) be an ultrafilter over \(\mathbb{A}\) that dominates the order filter. Define

\[ \mathcal{E}: \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}), \quad x \mapsto \text{WOT- lim}_{\alpha \to \mathcal{U}} \frac{1}{\|T_\alpha\|} |T_\alpha|(x). \]

The \(\|\mathcal{E}\| = 1\), \(\mathcal{E}|_{\mathcal{W}'} = \text{id}_{\mathcal{W}'}\), and \(\mathcal{E}(\mathcal{B}(\mathcal{H})) \subset \mathcal{W}'\).
GNoA’s for $C^*$-algebras, VI

1. $A$ is approximately amenable;
2. $A$ is pseudo-amenable;
3. $A$ is amenable.

Proof of 2. $\Rightarrow$ 3.

WLOG: $A$ is unital.

Via universal representation, we can suppose that $\text{id}_H \in A \subset B(H)$ such that $A^{\ast\ast} \sim_\varepsilon A''$. Hence, $A^{\ast\ast}$ is injective.
The following are equivalent for a C*-algebra $A$:

1. $A$ is approximately amenable;
2. $A$ is pseudo-amenable;
3. $A$ is amenable.

Proof of 2. $\Rightarrow$ 3.

WLOG: $A$ is unital. Via universal representation, we can suppose that $\text{id}_H \in A \subset B(H)$ such that $A^{**} \sim = A''$. Hence, $A^{**}$ is injective.
Corollary

The following are equivalent for a $C^*$-algebra $\mathcal{A}$:

1. $\mathcal{A}$ is approximately amenable;
2. $\mathcal{A}$ is pseudo-amenable;
3. $\mathcal{A}$ is amenable.

Proof of 2. $\Rightarrow$ 3.

WLOG: $\mathcal{A}$ is unital.

Via universal representation, we can suppose that $\text{id}_H \in \mathcal{A} \subset B(H)$ such that $\mathcal{A}^{**} \sim = A''$. Hence, $\mathcal{A}^{**}$ is injective.
Corollary

The following are equivalent for a $C^*$-algebra $\mathcal{A}$:

1. $\mathcal{A}$ is approximately amenable;

Proof of $2. \Rightarrow 3.$

WLOG: $\mathcal{A}$ is unital.

Via universal representation, we can suppose that $id_H \in \mathcal{A} \subset B(H)$ such that $\mathcal{A}^{\ast\ast} \sim = \mathcal{A}''$.

Hence, $\mathcal{A}^{\ast\ast}$ is injective.
The following are equivalent for a $C^*$-algebra $\mathcal{A}$:

1. $\mathcal{A}$ is approximately amenable;
2. $\mathcal{A}$ is pseudo-amenable;
Corollary

The following are equivalent for a $C^*$-algebra $\mathcal{A}$:

1. $\mathcal{A}$ is approximately amenable;
2. $\mathcal{A}$ is pseudo-amenable;
3. $\mathcal{A}$ is amenable.
GNoA’s for $C^*$-algebras, VI

Corollary

The following are equivalent for a $C^*$-algebra $\mathbb{A}$:

1. $\mathbb{A}$ is approximately amenable;
2. $\mathbb{A}$ is pseudo-amenable;
3. $\mathbb{A}$ is amenable.

Proof of 2. $\implies$ 3.

WLOG: $\mathbb{A}$ is unital.
Corollary

The following are equivalent for a C*-algebra $\mathcal{A}$:

1. $\mathcal{A}$ is approximately amenable;
2. $\mathcal{A}$ is pseudo-amenable;
3. $\mathcal{A}$ is amenable.

Proof of 2. $\implies$ 3.

WLOG: $\mathcal{A}$ is unital. Via universal representation,
Corollary

The following are equivalent for a $C^*$-algebra $\mathcal{A}$:

1. $\mathcal{A}$ is approximately amenable;
2. $\mathcal{A}$ is pseudo-amenable;
3. $\mathcal{A}$ is amenable.

Proof of 2. $\implies$ 3.

WLOG: $\mathcal{A}$ is unital. Via universal representation, we can suppose that $\text{id}_{\mathcal{H}} \in \mathcal{A} \subset \mathcal{B}(\mathcal{H})$
Corollary

The following are equivalent for a $C^*$-algebra $A$:

1. $A$ is approximately amenable;
2. $A$ is pseudo-amenable;
3. $A$ is amenable.

Proof of 2. $\implies$ 3.

WLOG: $A$ is unital. Via universal representation, we can suppose that $id_{\mathcal{H}} \in A \subset B(\mathcal{H})$ such that $A^{**} \cong A''$. 
Corollary

The following are equivalent for a $C^*$-algebra $\mathcal{A}$:

1. $\mathcal{A}$ is approximately amenable;
2. $\mathcal{A}$ is pseudo-amenable;
3. $\mathcal{A}$ is amenable.

Proof of 2. $\implies$ 3.

WLOG: $\mathcal{A}$ is unital. Via universal representation, we can suppose that $\text{id}_H \in \mathcal{A} \subset B(H)$ such that $\mathcal{A}^{**} \cong \mathcal{A}''$. Hence, $\mathcal{A}^{**}$ is injective.
Canadian Abstract Harmonic Analysis Symposium (CAHAS) 2020
Canadian Abstract Harmonic Analysis Symposium (CAHAS) 2020
In the honor of Anthony To-Ming Lau
Canadian Abstract Harmonic Analysis Symposium (CAHAS) 2020

In the honor of Anthony To-Ming Lau
on the occasion of his retirement
Canadian Abstract Harmonic Analysis Symposium (CAHAS) 2020

In the honor of Anthony To-Ming Lau

on the occasion of his retirement

Organizers:
Wittstock moduli of elementary operators and their application to generalized notions of amenability

Volker Runde

Wittstock moduli of elementary operators

Generalized notions of amenability

Paulsen’s off-diagonal technique

Conclusion

Advertisements

Advertisement I

Canadian Abstract Harmonic Analysis Symposium (CAHAS) 2020

In the honor of Anthony To-Ming Lau on the occasion of his retirement

Organizers:

Brian E. Forrest,
Canadian Abstract Harmonic Analysis Symposium (CAHAS) 2020

In the honor of Anthony To-Ming Lau on the occasion of his retirement

Organizers:
Brian E. Forrest, Volker Runde,
Canadian Abstract Harmonic Analysis Symposium (CAHAS) 2020

In the honor of Anthony To-Ming Lau on the occasion of his retirement

Organizers:
Brian E. Forrest, Volker Runde, Keith F. Taylor
Canadian Abstract Harmonic Analysis Symposium (CAHAS) 2020

In the honor of Anthony To-Ming Lau
on the occasion of his retirement

Organizers:
Brian E. Forrest, Volker Runde, Keith F. Taylor

To be held as a two-day workshop at BIRS in May/June
Canadian Abstract Harmonic Analysis Symposium (CAHAS) 2020

In the honor of Anthony To-Ming Lau
on the occasion of his retirement

Organizers:
Brian E. Forrest, Volker Runde, Keith F. Taylor

To be held as a two-day workshop at BIRS in May/June
at a date yet TBA.
Wittstock moduli of elementary operators and their application to generalized notions of amenability

Volker Runde

Wittstock moduli of elementary operators

Generalized notions of amenability

Paulsen’s off-diagonal technique

Conclusion

Advertisements
Soon to come to a bookstore near you:
Soon to come to a bookstore near you: