Jacobson’s lemma, exponential spectrum and homotopy theory

Thomas Ransford
Université Laval
Banach Algebras and Applications
University of Manitoba, Winnipeg, July 2019
Lemma

Let $R$ be a ring with 1, and let $a, b \in R$. Then $(1 - ab)$ is invertible in $R$ iff $(1 - ba)$ is invertible in $R$. 
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**Idea of proof (‘Desert-island formula’)**

$$(1 - ab)^{-1}$$
Jacobson’s lemma

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$$
Jacobson’s lemma in Banach algebras

Notation:

- $A$ = Banach algebra with 1.
- $G(A)$ = the group of invertible elements of $A$. This is an open subset of $A$.
- The spectrum of $a \in A$ is

$$\sigma(a) := \{ \lambda \in \mathbb{C} : \lambda 1 - a \notin G(A) \}.$$

Jacobson’s lemma for spectrum

If $a, b \in A$, then

$$\sigma(ab) \setminus \{0\} = \sigma(ba) \setminus \{0\}.$$
Denote by $G_1(A)$ the connected component of $G(A)$ containing $1$. It can be shown that

$$G_1(A) = \{ e^{a_1} e^{a_2} \cdots e^{a_n} : a_1, \ldots, a_n \in A, \ n \geq 1 \}.$$ 

The exponential spectrum of $a \in A$ is defined by

$$\epsilon(a) := \{ \lambda \in \mathbb{C} : \lambda 1 - a \notin G_1(A) \}.$$ 

It can be shown that $\epsilon(a)$ is a compact subset of $\mathbb{C}$ satisfying

$$\partial \epsilon(a) \subset \sigma(a) \subset \epsilon(a).$$
Problem: (Murphy, 1992) Do we always have

\[ \epsilon(ab) \setminus \{0\} = \epsilon(ba) \setminus \{0\}? \]

Equivalently, is it true that

\[ (1 - ab) \in G_1(A) \iff (1 - ba) \in G_1(A)? \]
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Results:

- Yes, if \( A \) commutative (trivial).
- Yes, if \( G(A) \) connected (trivial) (e.g. von Neumann algebras).
- Yes, if \( G(A) \) is dense in \( A \) (Murphy).
- Yes, if \( A \) is the Calkin algebra (Murphy).
- Yes, for certain algebras \( A \) resembling the Calkin algebra (Grobler–Raubenheimer).
- No, in general (Klaja–R).
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- **No,** in general (Klaja–R).
**Basic idea:** Take $A := C(X, B)$, where $X$ is compact and $B$ is a Banach algebra. Note that $G(A) = C(X, G(B))$ and that

$$f \in G_1(A) \iff f \text{ homotopic to 1 in } C(X, G(B)).$$
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- Need \( A \) non-commutative, so choose \( B \) non-commutative. We take \( B := M_2(\mathbb{C}) \).
- Need \( G(A) \) disconnected, so choose \( X \) non-contractible.
**Strategy for the counterexample**

**Basic idea:** Take $A := C(X, B)$, where $X$ is compact and $B$ is a Banach algebra. Note that $G(A) = C(X, G(B))$ and that

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**How to choose $X, B$?** We want them as simple as possible, but there are constraints.

- Need $A$ non-commutative, so choose $B$ non-commutative. We take $B := M_2(\mathbb{C})$.
- Need $G(A)$ disconnected, so choose $X$ non-contractible.
- Need $G(A)$ non-dense in $A$, so $X := S^1$ does not work. In fact we take $X := S^4$.

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We write

$$S^4 = \left\{ (z_1, z_2, z_3) \in \mathbb{C}^3 : |z_1|^2 + |z_2|^2 + |z_3|^2 = 1, \text{Im } z_3 = 0 \right\}.$$ 

**Theorem (Klaja–R)**

Let $A := C(S^4, M_2(\mathbb{C}))$, and define $a, b \in A$ by

$$a(z_1, z_2, z_3) := \frac{\sqrt{2}}{1 + iz_3} \begin{pmatrix} z_1 & 0 \\ z_2 & 0 \end{pmatrix},$$

$$b(z_1, z_2, z_3) := \frac{\sqrt{2}}{1 + iz_3} \begin{pmatrix} \bar{z}_1 & \bar{z}_2 \\ 0 & 0 \end{pmatrix}.$$ 

Then $(1 - ba) \in G_1(A)$, but $(1 - ab) \notin G_1(A)$. 

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Proof that \((1 - ba) \in G_1(A)\)

A calculation gives

\[
(1 - ba)(z_1, z_2, z_3) = \begin{pmatrix} \phi(z_3) & 0 \\ 0 & 1 \end{pmatrix},
\]

where \(\phi(z_3) := -\left(\frac{1 - iz_3}{1 + iz_3}\right)^2\). This factors as

\[
\mathbb{S}^4 \to [-1, 1] \to \mathbb{T} \to GL_2(\mathbb{C})
\]

\[
(z_1, z_2, z_3) \mapsto z_3 \mapsto \phi(z_3) \mapsto \begin{pmatrix} \phi(z_3) & 0 \\ 0 & 1 \end{pmatrix},
\]

and the middle map is null-homotopic. Hence \((1 - ba) \in G_1(A)\).
This time a calculation gives

\[(1 - ab)(z_1, z_2, z_3) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{2}{(1 + iz_3)^2} \begin{pmatrix} z_1\bar{z}_1 & z_1\bar{z}_2 \\ z_2\bar{z}_1 & z_2\bar{z}_2 \end{pmatrix} .\]
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\]

- Define \(f : S^4 \to S^3\) by \(f := p(1 - ab)/\|p(1 - ab)\|\), where \(p : M_2(\mathbb{C}) \to \mathbb{C}^2\) denotes projection onto second column.
- If \((1 - ab)\) were homotopic to 1 in \(C(S^4, GL_2(\mathbb{C}))\), then \(f\) would be null-homotopic in \(C(S^4, S^3)\).
This time a calculation gives

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- If \((1 - ab)\) were homotopic to 1 in \( C(S^4, GL_2(\mathbb{C}))\), then \( f \) would be null-homotopic in \( C(S^4, S^3)\).
- On the other hand, it is not hard to construct a homotopy in \( C(S^4, S^3) \) between \( f \) and \( Eh \), where \( Eh : S^4 \to S^3 \) is the suspension of the Hopf map \( h : S^3 \to S^2 \).
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- On the other hand, it is not hard to construct a homotopy in \(C(S^4, S^3)\) between \(f\) and \(Eh\), where \(Eh : S^4 \to S^3\) is the suspension of the Hopf map \(h : S^3 \to S^2\).
- It is known that \(Eh\) is not null-homotopic in \(C(S^4, S^3)\).
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This time a calculation gives

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- Define \(f : \mathbb{S}^4 \to \mathbb{S}^3\) by \(f := p(1 - ab)/\|p(1 - ab)\|\), where \(p : M_2(\mathbb{C}) \to \mathbb{C}^2\) denotes projection onto second column.
- If \((1 - ab)\) were homotopic to 1 in \(C(\mathbb{S}^4, GL_2(\mathbb{C}))\), then \(f\) would be null-homotopic in \(C(\mathbb{S}^4, \mathbb{S}^3)\).
- On the other hand, it is not hard to construct a homotopy in \(C(\mathbb{S}^4, \mathbb{S}^3)\) between \(f\) and \(Eh\), where \(Eh : \mathbb{S}^4 \to \mathbb{S}^3\) is the suspension of the Hopf map \(h : \mathbb{S}^3 \to \mathbb{S}^2\).
- It is known that \(Eh\) is \textbf{not} null-homotopic in \(C(\mathbb{S}^4, \mathbb{S}^3)\).
- Conclusion: \((1 - ab) \notin G_1(A)\).
(1) In fact the elements \( a \) and \( b \) above satisfy

\[
\epsilon(ab) = \{ \lambda \in \mathbb{C} : |\lambda - 1| \leq 1 \},
\]

\[
\epsilon(ba) = \{ \lambda \in \mathbb{C} : |\lambda - 1| = 1 \}.
\]
(1) In fact the elements $a$ and $b$ above satisfy

\[
\begin{align*}
\epsilon(ab) &= \{ \lambda \in \mathbb{C} : |\lambda - 1| \leq 1 \}, \\
\epsilon(ba) &= \{ \lambda \in \mathbb{C} : |\lambda - 1| = 1 \}.
\end{align*}
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(2) The algebra $A := \mathbb{C}(\mathbb{S}^4, M_2(\mathbb{C}))$ is a $\mathbb{C}^*$-algebra.
(1) In fact the elements $a$ and $b$ above satisfy 

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(2) The algebra $A := C(S^4, M_2(\mathbb{C}))$ is a C*-algebra.

**Question:** Do there exist a Banach space $E$ and bounded operators $S, T$ on $E$ such that

$$\epsilon(ST) \setminus \{0\} \neq \epsilon(TS) \setminus \{0\}?$$
Reference:
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Thank you for your attention!