Inversion problem in measure and Fourier-Stieltjes algebras

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(joint work with Mateusz Wasilewski)

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Basic definitions

Let $G$ be a locally compact Abelian group with the dual $\hat{G}$ and let $M(G)$ denote the Banach algebra of all complex-valued Borel regular measures on $G$ equipped with the total variation norm and the convolution product. The Gelfand space of $M(G)$ (the set of all multiplicative - linear functionals endowed with the weak* topology) will be abbreviated $\triangle(M(G))$. 

The Fourier-Stieltjes transform for $\gamma \in \hat{G}$ and $\mu \in M(G)$ we define $\hat{\mu}(\gamma) = \int_G \gamma(t) d\mu(t)$. Since the convolution is transferred to the pointwise product via Fourier-Stieltjes transform it is clear that $\hat{G}$ is canonically embedded into $\triangle(M(G))$. 

The Gelfand transform for $\mu \in M(G)$ we define the Gelfand transform of $\mu$ as a function $\hat{\mu} : \triangle(M(G)) \to \mathbb{C}$ given by the formula $\hat{\mu}(\phi) := \phi(\mu)$. We treat the Fourier-Stieltjes transform as the restriction of the Gelfand transform to $\hat{G}$. 

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I will discuss two problems proposed by N. Nikolski in the paper 'In search of the invisible spectrum'.

Let $\mu \in M(G)$ satisfy $\|\mu\| \leq 1$ and $|\hat{\mu}(\gamma)| \geq \delta$ for every $\gamma \in \hat{G}$ where $\delta > 0$ is fixed.

Problem 1 (qualitative)
What is the minimal $\delta_0 > 0$ such that for every $\delta > \delta_0$ the measure $\mu$ is automatically invertible?

Problem 2 (quantitative)
How can we estimate the norm of the inverse for $\delta > \delta_0$?

Remark
These problems are non-trivial because of the Wiener-Pitt phenomenon - the existence of non-invertible measures with Fourier-Stieltjes transforms bounded away from zero.
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The solution of Problem 1 (qualitative)

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Let $\mu \in \mathcal{M}(G)$ satisfy $\|\mu\| \leq 1$ and $|\hat{\mu}| \geq \delta > \frac{1}{2}$. Then $\mu$ is invertible.

Sharpness of the main theorem

This theorem is sharp for any non-discrete locally compact Abelian group: any continuous (non-atomic) probability measure with $\hat{\mu}(\hat{G}) \subset \mathbb{R}_+$ and $\sigma(\mu) = \overline{D}$ leads to the sharpness of this theorem - it is enough to consider $\nu := \frac{1}{2} \mu + \frac{1}{2} \delta_0$. Then $\|\nu\| = 1$, $|\hat{\nu}| \geq \frac{1}{2}$ but $\nu$ is not invertible as $0 \in \sigma(\nu) := \{ \lambda \in \mathbb{C} : \nu - \lambda \delta_0 \text{ is not invertible} \}$. 
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Partial solution of the second problem by N. Nikolski

Let $\mu \in M(G)$ satisfy $\|\mu\| \leq 1$ and $|\hat{\mu}| \geq \delta > \frac{1}{\sqrt{2}}$. Then $\mu$ is invertible and $\|\mu^{-1}\| \leq \frac{1}{2\delta^2 - 1}$.

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The proof is based on the following lemma.

Lemma

Let \( \mu \in M(G) \) satisfy \( \| \mu \| \leq 1 \) and let \( \mu = \lambda \delta_0 + \nu \) where \( \nu(\{0\}) = 0 \) and \( |\lambda| \geq \delta > \frac{1}{2} \). Then \( \mu \) is invertible and \( \| \mu^{-1} \| \leq \frac{1}{2\delta - 1} \).
Elementary fact

In order to improve the result of Nikolski we need first to prove an elementary fact.

\[
\text{Fact on real numbers}
\]

Let \((x_n)_{n=1}^{\infty}\) be a non-increasing sequence of positive real numbers satisfying:

\[
\sum_{n=1}^{\infty} x_n \leq 1,
\]

\[
\sum_{n=1}^{\infty} x_n^2 \geq \delta^2 > 1/4.
\]

Then \(x_1 \geq \delta/2\) and \(x_1 + x_2 \geq \delta\).
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Then

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x_1 \geq \delta^2 \text{ and } \quad x_1 + x_2 \geq \delta.
\]
Proof of the elementary fact

First part
To prove the first inequality we simply observe that
\[ \delta^2 \leq \sum_{n=1}^{\infty} x_n^2 \leq x_1 \cdot \sum_{n=1}^{\infty} x_n \leq x_1. \]

Second part
For the second one we proceed as follows:
\[ \delta^2 \leq \sum_{n=1}^{\infty} x_n^2 = x_1^2 + \sum_{n=2}^{\infty} x_n^2 \leq x_1^2 + x_2 \cdot \sum_{n=2}^{\infty} x_n \leq x_1^2 + x_2 \cdot \left(1 - x_1\right). \]
If \( x_1 \geq \delta \) than we are done and if \( x_1 < \delta \) we are allowed to write the above inequality as \( x_2 \geq \delta^2 - x_1^2 - x_1 \cdot x_1 \). Thus it is enough to verify the quadratic inequality
\[ x_1 + \delta^2 - x_2 ^2 - x_2 \cdot x_1 \geq \delta \text{ for } x_1 \in [\delta^2, \delta). \]
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If \( x_1 \geq \delta \) than we are done and if \( x_1 < \delta \) we are allowed to write the above inequality as \( x_2 \geq \frac{\delta^2 - x_1^2}{1-x_1} \). Thus it is enough to verify the quadratic inequality

\[ x_1 + \frac{\delta^2 - x_1^2}{1-x_1} \geq \delta \text{ for } x_1 \in [\delta^2, \delta). \]
Two largest coefficients

Now, we can prove the following theorem which is crucial to proceed further.

Let $\mu \in M(G)$ satisfy $\|\mu\| \leq 1$, $|\hat{\mu}| \geq \delta > \frac{1}{2}$ and let $\mu_d = \sum_{n=1}^{\infty} a_n \tau_n$, where $|a_1| \geq |a_2| \geq ...$ and $\tau_n \in G$.

Then $|a_1| \geq \delta^2$ and $|a_1| + |a_2| \geq \delta$. 
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On two largest coefficients

Let $\mu \in M(G)$ satisfy $\|\mu\| \leq 1$, $|\hat{\mu}| \geq \delta > \frac{1}{2}$ and let

$$\mu_d = \sum_{n=1}^{\infty} a_n \delta_{\tau_n}, \text{ where } |a_1| \geq |a_2| \geq \ldots \text{ and } \tau_n \in G.$$ 

Then $|a_1| \geq \delta^2$ and $|a_1| + |a_2| \geq \delta$. 
Proof of the fact on two largest coefficients

We check the assumptions of the lemma.
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This is clear as \( \sum_{n=1}^{\infty} |a_n| = \|\mu_d\| \leq \|\mu\| \leq 1 \)
Proof of the fact on two largest coefficients

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**First assumption**

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**Second assumption**

By G-W theorem \((\hat{\mu}_d(\hat{G} \subset \hat{\mu}(\hat{G}))\) we have \(|\hat{\mu}_d| \geq \delta\) and since \(\hat{G}\) is dense in \(b\hat{G} = \hat{G}_d\) we get from Parseval’s identity:

\[
\|\mu_d\|_{L^2(G_d)}^2 = \sum_{n=1}^{\infty} |a_n|^2 = \int_{b\hat{G}} |\hat{\mu}_d|^2 \, dx \geq \delta^2.
\]
The main theorem

Let $\mu \in M(G)$ satisfy $\|\mu\| \leq 1$ and $|\hat{\mu}(\gamma)| > \delta > 1$ for every $\gamma \in \hat{G}$. Let $\mu_d = \sum_{n=1}^{\infty} a_n \delta^{\tau_n}$, $|a_1| \geq |a_2| \geq \ldots$

If the order of element $\tau_2 - \tau_1$ is infinite then $|a_1| \geq 1 - \delta + \sqrt{17 \delta^2 + 6 \delta - 7} \geq 3 \delta - \frac{1}{2}$ and $\|\mu - 1\| \leq \frac{1}{3} \delta - \frac{2}{3}$ for $\delta > \frac{2}{3}$, $\|\mu - 1\| \leq 2 - (1 + \delta) + \sqrt{17 \delta^2 + 6 \delta - 7}$ for $\delta > -1 + \sqrt{\frac{33}{8}} \approx 0.593$. 

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The main theorem

Let $\mu \in M(G)$ satisfy $\|\mu\| \leq 1$ and $|\hat{\mu}(\gamma)| > \delta > \frac{1}{2}$ for every $\gamma \in \hat{G}$. Let

$$\mu_d = \sum_{n=1}^{\infty} a_n \delta_{\tau_n}, \quad |a_1| \geq |a_2| \geq \ldots.$$

If the order of element $\tau_2 - \tau_1$ is infinite then

$$|a_1| \geq \frac{1 - \delta + \sqrt{17\delta^2 + 6\delta - 7}}{4} \geq \frac{3}{2} \delta - \frac{1}{2} \quad \text{and}$$

$$\|\mu^{-1}\| \leq \frac{1}{3\delta - 2} \quad \text{for} \ \delta > \frac{2}{3},$$

$$\|\mu^{-1}\| \leq \frac{2}{-(1 + \delta) + \sqrt{17\delta^2 + 6\delta - 7}} \quad \text{for} \ \delta > \frac{-1 + \sqrt{33}}{8} \approx 0.593.$$
Proof of the main theorem

We consider the following measure $\nu := c \mu \ast \delta_{-\tau_1}$ where $ca_1 = |a_1|$ and observe that we are allowed to work with $\nu$ instead of $\mu$. Then

$$\nu_d = |a_1| \delta_0 + ca_2 \delta_{\tau_2 - \tau_1} + \rho,$$

where $\rho := c \sum_{n=3}^{\infty} a_n \delta_{\tau_n - \tau_1}$. 

As $\tau_2 - \tau_1$ has infinite order we pick a sequence $\gamma_n \in \hat{G}$ such that $\hat{\delta}_{\tau_2 - \tau_1}(\gamma_n)ca_2 \to -|a_2|$. By G-W theorem:

$$||a_1| + ca_2 \hat{\delta}_{\tau_2 - \tau_1}(\gamma_n) + \hat{\rho}(\gamma_n)|| \geq \delta.$$

Of course, $||\rho|| = ||\mu_d|| - |a_1| - |a_2|$ and passing with $n$ to infinity we get $||\mu_d|| - 2|a_2| \geq \delta$. 
The inequality $\|\mu_d\| - 2|a_2| \geq \delta$ is equivalent to $|a_2| \leq \frac{\|\mu_d\| - \delta}{2} \leq \frac{1 - \delta}{2}$. Since $|a_1| + |a_2| \geq \delta$, we obtain $|a_1| \geq \frac{3}{2}\delta - \frac{1}{2}$ which proves the first assertion.

In order to get a more refined bound we recall from the proof of the elementary fact that:

$$|a_2| \geq \frac{\delta^2 - |a_1|^2}{1 - |a_1|}.$$ 

As $|a_2| \leq \frac{1 - \delta}{2}$ we get (again) a quadratic inequality on $|a_1|$ whose verification finishes the proof of the theorem.
It is worth to state the full formulation for the most classical case of $G = \mathbb{Z}$. 

$\|f\|_{A(T)} \leq 1$ and $|f(t)| \geq \delta > \frac{1}{2}$ for every $t \in T$. Then $\|f\|_{A(T)} \leq \frac{1}{3} \delta - \frac{2}{\delta + 2} + \frac{\sqrt{17}}{\delta^{2}} + \frac{6}{\delta} - \frac{7}{\delta - 1} \approx 0.593$. 

Ohrysko (Chalmers)
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Let $f \in A(\mathbb{T})$ satisfy $\|f\|_{A(\mathbb{T})} \leq 1$ and $|f(t)| \geq \delta > \frac{1}{2}$ for every $t \in \mathbb{T}$. Then

$$\left\| \frac{1}{f} \right\|_{A(\mathbb{T})} \leq \frac{1}{3\delta - 2} \quad \text{for } \delta > \frac{2}{3},$$

$$\left\| \frac{1}{f} \right\|_{A(\mathbb{T})} \leq \frac{2}{-(1 + \delta) + \sqrt{17\delta^2 + 6\delta - 7}} \quad \text{for } \delta > \frac{-1 + \sqrt{33}}{8} \approx 0.593.$$
As it was shown before $\delta_0 = \frac{1}{2}$ is a critical constant for invertibility on any non-discrete group. But for discrete case we have the following result due to N. Nikolsky:
Remarks on the condition $\delta > \frac{1}{2}$

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**Discrete groups**

Let $G$ be a an infinite discrete group and let $\delta \leq \frac{1}{2}$. Then

$$\sup\{\|\mu^{-1}\| : \mu \in M(G), \|\mu\| \leq 1, |\hat{\mu}| \geq \delta\} = \infty.$$
Fourier-Stieltjes algebras

Let $G$ be an arbitrary locally compact group (not necessarily Abelian) and let $B(G)$ denote the Fourier-Stieltjes algebra of $G$. In this setting the first problem of Nikolski can be formulated as follows.

**Qualitative problem for Fourier-Stieltjes algebras**

Let $f \in B(G)$ satisfy $\|f\| \leq 1$ and $|f(x)| \geq \delta > \delta_0$ for every $x \in G$. What is the minimal value of $\delta_0$ assuring the invertibility of $f$?
(Weak) almost periodicity

A continuous bounded function on $G$ is called (weakly) almost periodic if the set of all of its translates is precompact in (weak topology) uniform topology, respectively. The set of all such functions will be denoted by $AP(G)$ ($WAP(G)$).

A classical result of Eberlain states that $B(G) \subset WAP(G)$. Moreover, there exists an invariant mean $M \in (WAP(G))^*$ and restricting this mean to $B(G)$ we obtain the following decomposition:

$B(G) = B_c(G) \oplus (B(G) \cap AP(G))$ where

$B_c(G) := \{ f \in B(G) : M(|f|) = 0 \}$. 

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Main result for Fourier-Stieltjes algebras

The solution of the qualitative problem is a direct analogue of the result for measure algebras.

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The solution of the qualitative problem is a direct analogue of the result for measure algebras.

Solution of the qualitative problem for Fourier-Stieltjes algebras

Let \( f \in B(G) \) satisfy \( \|f\| \leq 1 \) and \( |f(x)| > \frac{1}{2} \) for every \( x \in G \). Then \( f \) is invertible.

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Glicksberg-Wik theorem for Fourier-Stieltjes algebras

Let $f \in B(G)$ have the decomposition $f = g + h$ with $g \in B(G) \cap AP(G)$ and $h \in B_c(G)$. Then $g(G) \subset \overline{f(G)}$. 
Proof of G-W theorem for Fourier-Stieltjes algebras

Let us fix $\varepsilon > 0$. Without loss of generality, it is enough to show $g(e) \in \overline{f(G)}$. First we prove the existence of distinct group elements $(x_n)_{n=1}^\infty$ such that:

$$\left| h(x_1) \right| < \varepsilon \quad \text{and} \quad \left| h(x_n - x_{n-1}) \right| < \varepsilon \quad \text{for} \quad j < n \quad \text{and} \quad n > 1.$$ 

Since $h \in B_c(G)$ we clearly have $\inf_{x \in G} |h(x)| = 0$ and we can choose $x_1$ easily.

Suppose that we have already picked up $x_1, \ldots, x_{n-1}$. 

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Proof of G-W theorem for Fourier-Stieltjes algebras

Let us fix $\varepsilon > 0$. Without loss of generality, it is enough to show $g(e) \in \overline{f(G)}$. First we prove the existence of distinct group elements $(x_n)_{n=1}^{\infty}$ such that:

**Special sequence**

$|h(x_1)| < \varepsilon$ and $|h(x_n x_j^{-1})| < \varepsilon$ for $j < n$ and $n > 1$.

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Suppose that we have already picked up $x_1, \ldots, x_{n-1}$.
Proof continued

We consider an auxiliary function $u \in B_c(G)$.

As $u \in B_c(G)$ we also have $\inf_{x \in G} |u(x)| = 0$ so we are allowed to choose $x_n$ different from $x_1, \ldots, x_{n-1}$ or $|h(e)| < \varepsilon$ and the argument is finished. Let us define the following set of functions in $B(G) \cap AP(G)$:
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We consider an auxiliary function $u \in B_c(G)$.

**Auxiliary function**

$$u(x) = \sum_{j=1}^{n-1} |h(xx_j^{-1})|^2.$$  

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Let us define the following set of functions in \( B(G) \cap AP(G) \):

**Set of translates**

\[ X = \{ g_{x_n} : n \in \mathbb{N} \} \text{ where } g_x(y) = g(xy) \text{ for } x, y \in G. \]
Proof continued 2

By definition, this set is precompact in the uniform topology. Hence there exists a subsequence \((g_{x_n^k})_{k \in \mathbb{N}}\) which is a Cauchy sequence:

In particular,

Finally,
Proof continued 2

By definition, this set is precompact in the uniform topology. Hence there exists a subsequence \((g_{x_{n_k}})_{k \in \mathbb{N}}\) which is a Cauchy sequence:

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\|g_{x_{n_{k+1}}} - g_{x_{n_k}}\|_\infty < \varepsilon \text{ for } k > N.
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**Cauchy sequence**

\[\|g_{x_{n_k+1}} - g_{x_{n_k}}\|_\infty < \varepsilon \text{ for } k > N.\]

In particular,

**Inequality**

\[|g_{x_{n_k+1}}(x_n^{-1}) - g_{x_{n_k}}(x_n^{-1})| = |g(x_{n_k+1}x_n^{-1}) - g(e)| < \varepsilon.\]

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Proof continued 2

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**Cauchy sequence**

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\]

Finally,

**Final inequality**

\[
|f(x_{n_k+1}^{-1}x_{n_k}) - g(e)| = |h(x_{n_k+1}^{-1}x_{n_k}) + g(x_{n_k+1}x_{n_k}^{-1}) - g(e)| < 2\varepsilon.
\]
Let $f \in B(G)$ satisfy $\|f\| \leq 1$, $\inf_{x \in G} |f(x)| > \frac{1}{2}$ and let $f = g + h$ with $g \in AP(G) \cap B(G)$ and $h \in B_c(G)$. By the variant of G-W theorem for Fourier-Stieltjes algebras we have $\inf_{x \in G} |g(x)| > \frac{1}{2}$ and since $G$ is dense in $\Delta(A(P(G) \cap B(G))$ we obtain $|\varphi(g)| > \frac{1}{2}$ for every $\varphi \in \Delta(A(P(G) \cap B(G))$. Suppose now that $f$ is not invertible. Then there exists $\varphi_0 \in \Delta(B(G))$ such that $\varphi_0(f) = 0$. This gives $|\varphi_0(g)| = |\varphi_0(h)|$. But $|\varphi_0(g)| > \frac{1}{2}$ and $|\varphi_0(h)| \leq \|h\| \leq 1 - \|g\| \leq 1 - r(g) < \frac{1}{2}$ which is a contradiction.
Let $f \in B(G)$ satisfy $\|f\| \leq 1$ and $\inf_{x \in G} |f(x)| \geq \delta > \frac{1}{\sqrt{2}}$. Then $f$ is invertible and $\|f^{-1}\|_{B(G)} \leq \frac{1}{2\delta^2 - 1}$. 
Quantitative result

Generalization of Nikolski’s result

Let \( f \in B(G) \) satisfy \( \| f \| \leq 1 \) and \( \inf_{x \in G} |f(x)| \geq \delta > \frac{1}{\sqrt{2}} \). Then \( f \) is invertible and \( \| f^{-1} \|_{B(G)} \leq \frac{1}{2\delta^2 - 1} \).

Proof

We consider \( |f|^2 = f \cdot \bar{f} \). One can show that \( |f|^2 = m(|f|^2)1 + f_0 \). Since \( |f|^2 \geq \delta^2 > \frac{1}{2} \) we also have \( m(|f|^2) \geq \delta^2 > \frac{1}{2} \) and we are allowed to write down the inverse as an absolutely convergent series.
Thank You for your attention!