Linking the boundary and exponential spectra via the restricted topology

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Outline of talk

The exponential spectrum

The boundary spectrum

Motivation: The "thin" and "fat" boundaries

Notation and preliminaries

The restricted topology

Examples

The $\omega$-spectrum

A chain of boundaries

A chain of connected hulls

Final remarks and an application
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The boundary spectrum
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The exponential spectrum

Let $A$ be a complex Banach algebra with unit $1$ and $A^{-1}$ the set of all invertible elements of $A$. 

$\text{Exp}(A) = \{ e^{a_1} e^{a_2} \ldots e^{a_n} : n \in \mathbb{N}, a_1, a_2, \ldots, a_n \in A \} = \text{Comp}_A(1, A^{-1})$

The exponential spectrum of $a \in A$ is defined as $\epsilon(a) = \{ \lambda \in \mathbb{C} : a - \lambda 1/\lambda \in \text{Exp}(A) \}$.

If $\sigma(a)$ denotes the spectrum of $a$ and $\eta_{\sigma}(a)$ the connected hull of $\sigma(a)$, then $\sigma(a) \subseteq \epsilon(a) \subseteq \eta_{\sigma}(a)$.

$\epsilon(a)$ is non-empty and compact.
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The *exponential spectrum* of $a \in A$ is defined as

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$$\omega_T(a) = \cap \{\sigma(a + c) : c \in N(T)\}.$$
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Note that

$$\sigma(Ta) \subseteq \omega_T(a) \subseteq \sigma(a).$$
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Robin Harte used the exponential spectrum to prove:

**Theorem**

If $T : A \to B$ is bounded and has closed range, then

$$\eta\omega_T(a) = \eta\sigma(Ta)$$ for all $a \in A$. 

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The key result: If $T$ is bounded and onto, then 
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*Mouton, Harte* Boundary and exponential spectra, restricted topology
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Let $A$ be a (complex, unital) Banach algebra and $S$ the set of all non-invertible elements of $A$. The boundary spectrum of $a \in A$ is defined as

$$S_{\partial}(a) = \{ \lambda \in \mathbb{C} : a - \lambda 1 \in \partial S \} = \{ \lambda \in \mathbb{C} : (a - \lambda 1)/\in A - 1 \cup \text{int } S \}.$$ 

where $\text{int } S$ denotes the topological interior of the set $S$.

If $\partial \sigma(a)$ denotes the topological boundary of $\sigma(a)$, then

$$\partial \sigma(a) \subseteq S_{\partial}(a) \subseteq \sigma(a).$$

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Mouton, Harte | Boundary and exponential spectra, restricted topology
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$S_{\partial}(a)$ is non-empty and compact.
The spectral radius function is, in general, not continuous on the set of all positive elements in an ordered Banach algebra.

Theorem
Let $A$ be an ordered Banach algebra with algebra cone $C$ closed and normal. If $a \in C$ such that $S_{\partial}(a) \cap \mathbb{R}^+ = \{r(a)\}$ (where $r(a)$ is the spectral radius of $a$), then $r|_C$ is continuous at $a$.

The key result: the map $a \mapsto T(a) = \{\lambda \in C : |\lambda| \in S_{\partial}(a)\}$.


Mouton, Harte Boundary and exponential spectra, restricted topology
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The boundary spectrum was used to prove the following result about spectral continuity in ordered Banach algebras:

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The boundary spectrum was used to prove the following result about spectral continuity in ordered Banach algebras:

**Theorem**

Let $A$ be an ordered Banach algebra with algebra cone $C$ closed and normal. If $a \in C$ such that $S_\theta(a) \cap \mathbb{R}^+ = \{ r(a) \}$ (where $r(a)$ is the spectral radius of $a$), then $r|_C$ is continuous at $a$.

The key result: the map $a \mapsto T(a)$ is upper semicontinuous, where

$$T(a) = \{ \lambda \in \mathbb{C} : |\lambda| \in S_\theta(a) \}.$$
The boundary spectrum

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Recall: If $A$ is a Banach algebra and $a \in A$, then 
$$\partial \sigma(a) \subseteq S \partial(a) \subseteq \sigma(a).$$

$\partial \sigma(a)$: the “thin” boundary of $\sigma(a)$

$S \partial(a)$: the “fat” boundary of $\sigma(a)$

Using closed subalgebras $B$, it is possible to define a topology on $A$ in such a way that a whole range of “boundaries” can be obtained, with $B = C$ giving the “thin” boundary and $B = A$ the “fat” boundary of $\sigma(a)$. 

Mouton, Harte
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Corresponding to the “thin” and “fat” boundaries:

Is $\varepsilon(a)$ the “little” connected hull of $\sigma(a)$?

Is $\eta\sigma(a)$ the “big” connected hull of $\sigma(a)$?
Let $X$ be a topological space, $K \subseteq X$, $a \in X$ and $t \in X \setminus K$. Then:

- $\text{cl}_X(K)$: the closure of $K$
- $\text{Int}_X(K)$: the interior of $K$
- $\partial X(K)$: the boundary of $K$
- $\text{Nbd}_X(a)$: the collection of all neighbourhoods of $a$
- $\text{Comp}_X(t, X \setminus K)$: the (connected) component of $t$ in $X \setminus K$
- $\eta_t(K) = X \setminus \text{Comp}_X(t, X \setminus K)$: the connected hull relative to $t$ of $K$.
Notation and preliminaries

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collection of all neighbourhoods of $a$ \hspace{1cm} \text{Nbd}_X(a)

connected component of $t$ in $X \setminus K$ \hspace{1cm} \text{Comp}_X(t, X \setminus K)

connected hull relative to $t$ of $K$ \hspace{1cm} \eta_t(K) = X \setminus \text{Comp}_X(t, X \setminus K)$
The restricted topology

Recall: If $Y$ is a subset of a topological space $X$ then the relative topology of $Y$ induced by the topology of $X$ is obtained via the closure operation, for $K \subseteq Y$, as follows:

$$\text{cl}_Y(K) = Y \cap \text{cl}_X(K)$$

**Definition**

Let $B$ be a subgroup of an additive topological group $A$. If $K \subseteq A$ is arbitrary then its restricted closure in $A$ relative to $B$ is given by

$$\text{cl}_B(K) = \{ a \in A : \forall U \in \text{Nbd}_B(0) : (a - U) \cap K \neq \emptyset \}.$$
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The restricted closure does, in fact, define a topology:

1. $\text{cl}_B(\emptyset) = \emptyset$

2. If $K \subseteq H = \Rightarrow \text{cl}_B(K) \subseteq \text{cl}_B(H)$

3. $K \subseteq \text{cl}_B(K)$

4. $\text{cl}_B \text{cl}_B(K) \subseteq \text{cl}_B(K)$

5. $\text{cl}_B(K \cup H) \subseteq \text{cl}_B(K) \cup \text{cl}_B(H)$.
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**Theorem**

*If $B$ is a subgroup of a topological group $A$ and if $K, H \subseteq A$ then*

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*If $B = A$ the $B$-topology coincides with the topology of $A$.***

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**restricted topology or $B$-topology**

If $K \subseteq B$ then $\text{cl}^B(K) = \text{cl}_B(K).$

If $B = A$ the $B$-topology coincides with the topology of $A.$
Let \( B \) be a subgroup of a topological group \( A \), \( K \subseteq A \) and \( a \in A \). Then:

- \( \text{Int} B(K) \): the restricted interior of \( K \)
- \( \partial B(K) \): the restricted boundary of \( K \)
- \( \text{Nbd} B(a) \): the collection of all restricted neighbourhoods of \( a \)
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Examples

$C^0[0,1]$: Banach space of all continuous complex valued functions on $[0,1]$. 

$C^x$: the subset of $C^0[0,1]$ of all complex valued homogeneous polynomials on $[0,1]$ of degree 1.

Example: If $A = C^0[0,1]$, $B = C^0$ and $K = C^x$, then $cl_B(K) = K = cl_A(K)$ and $Int_A(K) = \emptyset = Int_B(K)$.
$C_\mathbb{C}[0,1]$: Banach space of all continuous complex valued functions on $[0,1]$ with the supremum norm
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$C_x$: the subset of $C_C[0, 1]$ of all complex valued homogeneous polynomials on $[0, 1]$ of degree 1

Example

If $A = C_C[0, 1]$, $B = \mathbb{C}$ and $K = C_x$, then $\text{cl}^B(K) = K = \text{cl}^A(K)$ and $\text{Int}^A(K) = \emptyset = \text{Int}^B(K)$. 
Examples

$C_\mathbb{R}[0, 1]$: the Banach space of all continuous real valued functions on $[0, 1]$ with the supremum norm

$P_\mathbb{R}[0, 1]$: the subset of $C_\mathbb{R}[0, 1]$ of all real valued polynomials on $[0, 1]$

Example

If $A = C_\mathbb{R}[0, 1]$, $B = \mathbb{R}$ and $K = P_\mathbb{R}[0, 1]$, then $\text{cl}^B(K) = K$, $\text{cl}^A(K) = A$, $\text{Int}^A(K) = \emptyset$ and $\text{Int}^B(K) = K$. 

Mouton, Harte
Boundary and exponential spectra, restricted topology
Example

Let $A = M_2^u(\mathbb{C})$, the space of all $2 \times 2$ complex upper triangular matrices, $B = \mathbb{C}, K = \left\{ \begin{pmatrix} w & z \\ 0 & w \end{pmatrix} : z \in \mathbb{C}, w \in \mathbb{Q} + i\mathbb{Q} \right\}$ and $S = \left\{ \begin{pmatrix} w & z \\ 0 & w \end{pmatrix} : z, w \in \mathbb{C} \right\}$. Then

$$\text{cl}^B(K) = S = \text{cl}^A(K)$$

and

$$\text{Int}^A(K) = \emptyset = \text{Int}^B(K).$$
The $\omega$-spectrum

**Definition**

Let $A$ be a topological algebra and $B$ a subalgebra of $A$ with unit $e$. If a map $\omega: A \rightarrow 2^B$ satisfies

$$\forall (a,b) \in A \times B: \omega(a-b) = \omega(a) - b \subseteq B,$$

set $H_{\omega} = \{a \in A: 0 \not\in \omega(a)\}$.

Then $\omega(a) = \{b \in B: a-b \not\in H_{\omega}\}$.

**Motivation**

If $A$ is a complex Banach algebra with unit $1$ and $B = \mathbb{C}$, then we can take $e = 1$ and $\omega = \sigma$; and then $H_{\sigma} = A - 1$.

Mouton, Harte

Boundary and exponential spectra, restricted topology
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If $A$ is a complex Banach algebra with unit $1$ and $B = \mathbb{C}$, then we can take $e = 1$ and $\omega = \sigma$; and then $H_\sigma = A - 1$. 

(Mouton, Harte, Boundary and exponential spectra, restricted topology)
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Motivation

If $A$ is a complex Banach algebra with unit 1 and $B = \mathbb{C}$, then we can take $e = 1$ and $\omega = \sigma$; and then $H_\sigma = A^{-1}$.  

Theorem

Let $A$ be a topological algebra, $B$ a subalgebra of $A$ with unit $e$ and $\omega : A \mapsto 2^B$ a map satisfying $\omega(a - b) = \omega(a) - b$ for all $(a, b) \in A \times B$. Then

$$\partial B \omega(a) = \{ b \in B : a - b \in \partial B (A \setminus H \omega) \}.$$ 

Let $A$ be a complex Banach algebra with unit $1$ and $a \in A$. If $B = C, e = 1$ and $\omega = \sigma$ (so $H \sigma = A \setminus 1$), then we have:

$$\partial C(a) := \partial C \sigma(a) = \{ \lambda \in C : a - \lambda 1 \in \partial C(A \setminus A - 1) \}.$$ 

Therefore we define, for any closed subalgebra $B$ of $A$ such that $1 \in B$:

$$\partial B(a) = \{ \lambda \in C : a - \lambda 1 \in \partial B(A \setminus A - 1) \}.$$
Theorem

Let $A$ be a topological algebra, $B$ a subalgebra of $A$ with unit $e$ and $\omega : A \mapsto 2^B$ a map satisfying $\omega(a - b) = \omega(a) - b$ for all $(a, b) \in A \times B$. Then

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Let $A$ be a complex Banach algebra with unit $1$ and $a \in A$. If $B = \mathbb{C}$, $e = 1$ and $\omega = \sigma$ (so $H_\sigma = A^{-1}$), then we have:
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Let $A$ be a topological algebra, $B$ a subalgebra of $A$ with unit $e$ and $\omega : A \mapsto 2^B$ a map satisfying $\omega(a - b) = \omega(a) - b$ for all $(a, b) \in A \times B$. Then

$$\partial^B \omega(a) = \{ b \in B : a - b \in \partial^B(A \setminus H_\omega) \}.$$

Let $A$ be a complex Banach algebra with unit 1 and $a \in A$. If $B = \mathbb{C}$, $e = 1$ and $\omega = \sigma$ (so $H_\sigma = A^{-1}$), then we have:

$$\partial^\mathbb{C} \sigma(a) = \{ \lambda \in \mathbb{C} : a - \lambda 1 \in \partial^\mathbb{C}(A \setminus A^{-1}) \}.$$
A chain of boundaries

**Theorem**

Let $A$ be a topological algebra, $B$ a subalgebra of $A$ with unit $e$ and $\omega : A \mapsto 2^B$ a map satisfying $\omega(a - b) = \omega(a) - b$ for all $(a, b) \in A \times B$. Then

$$\partial^B \omega(a) = \{ b \in B : a - b \in \partial^B(A \setminus H_\omega) \}.$$

Let $A$ be a complex Banach algebra with unit $1$ and $a \in A$. If $B = \mathbb{C}$, $e = 1$ and $\omega = \sigma$ (so $H_\sigma = A^{-1}$), then we have:

$$\partial_\mathbb{C}(a) := \partial^\mathbb{C} \sigma(a) = \{ \lambda \in \mathbb{C} : a - \lambda 1 \in \partial^\mathbb{C}(A \setminus A^{-1}) \}.$$
**Theorem**

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A chain of boundaries

**Theorem**

Let $A$ be a complex Banach algebra with unit 1. Then

1. $\partial_C(a) = \partial\sigma(a)$
A chain of boundaries

**Theorem**

Let $A$ be a complex Banach algebra with unit $1$. Then

1. $\partial_{\mathbb{C}}(a) = \partial \sigma(a)$ and
2. $\partial_{A}(a) = S_{\theta}(a),$

for all $a \in A$. 

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Boundary and exponential spectra, restricted topology
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Let $A$ be a complex Banach algebra with unit $1$. Then

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Let $A$ be a complex Banach algebra with unit $1$ and let $B$ be a closed subalgebra of $A$ such that $1 \in B$. Then

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Mouton, Harte

Boundary and exponential spectra, restricted topology
Let $B$ be a subgroup of a topological group $A$, $K \subseteq A$ and $t \in A \setminus K$. Then:

$\text{Comp}_B(t, A \setminus K)$: the restricted component of $t$ in $A \setminus K$

$\eta^B_t(K)$ = $A \setminus \text{Comp}_B(t, A \setminus K)$: the restricted connected hull relative to $t$ of $K$

If $K \subseteq B$ and $t \in B \setminus K$, then:

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A chain of connected hulls

**Notation**

Let $B$ be a subgroup of a topological group $A$, $K \subseteq A$ and $t \in A \setminus K$. Then:

$\eta_B^t(K) = A \setminus \text{Comp}_B(t, A \setminus K)$: the restricted component of $t$ in $A \setminus K$.

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Notation

Let $A$ be a complex normed algebra with unit $1$, $B$ a closed subalgebra of $A$ with $e = 1 \in B$ and $\omega : A \to 2^B$ a map satisfying $\omega(a - b) = \omega(a) - b$ for all $(a, b) \in A \times B$. 

Suppose that, for some $a \in A$, $\omega(a)$ is bounded. Then:

$\text{Comp}_{B}(\infty, B \setminus \omega(a))$: the unique unbounded restricted component of $B \setminus \omega(a)$

$\eta_{B \omega(a)} := B \setminus \text{Comp}_{B}(\infty, B \setminus \omega(a))$: the restricted connected hull of $\omega(a)$

Suppose that $1 \in \mathcal{H}_{\omega}$. Then:

$\text{Comp}_{B}(1, \mathcal{H}_{\omega})$: the restricted component of $1$ in $\mathcal{H}_{\omega}$

$\eta_{A \setminus \mathcal{H}_{\omega}} := A \setminus \text{Comp}_{B}(1, \mathcal{H}_{\omega})$: the restricted connected hull of $A \setminus \mathcal{H}_{\omega}$
Notation

Let $A$ be a complex normed algebra with unit $1$, $B$ a closed subalgebra of $A$ with $e = 1 \in B$ and $\omega : A \to 2^B$ a map satisfying $\omega(a - b) = \omega(a) - b$ for all $(a, b) \in A \times B$.

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A chain of connected hulls

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Let $A$ be a complex normed algebra with unit $1$, $B$ a closed subalgebra of $A$ with $e = 1 \in B$ and $\omega : A \to 2^B$ a map satisfying $\omega(a - b) = \omega(a) - b$ for all $(a, b) \in A \times B$.

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Let $A$ be a complex normed algebra with unit $1$, $B$ a closed subalgebra of $A$ with $e = 1 \in B$ and $\omega : A \to 2^B$ a map satisfying $\omega(a - b) = \omega(a) - b$ for all $(a, b) \in A \times B$.

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  \[ \text{Comp}^B(\infty, B \setminus \omega(a)) : \text{the unique unbounded restricted component of } B \setminus \omega(a) \]
  \[ \eta^B\omega(a) := B \setminus \text{Comp}^B(\infty, B \setminus \omega(a)) : \text{the restricted connected hull of } \omega(a) \]

- Suppose that $1 \in H_\omega$. Then:
A chain of connected hulls

**Notation**

Let $A$ be a complex normed algebra with unit $1$, $B$ a closed subalgebra of $A$ with $e = 1 \in B$ and $\omega : A \to 2^B$ a map satisfying $\omega(a - b) = \omega(a) - b$ for all $(a, b) \in A \times B$.

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  $\text{Comp}^B(\infty, B \setminus \omega(a))$: the unique unbounded restricted component of $B \setminus \omega(a)$

  $\eta^B \omega(a) := B \setminus \text{Comp}^B(\infty, B \setminus \omega(a))$: the *restricted connected hull* of $\omega(a)$

- Suppose that $1 \in H_\omega$. Then:

  $\text{Comp}^B(1, H_\omega)$: the restricted component of $1$ in $H_\omega$
A chain of connected hulls

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Let $A$ be a complex normed algebra with unit $1$, $B$ a closed subalgebra of $A$ with $e = 1 \in B$ and $\omega : A \to 2^B$ a map satisfying $\omega(a - b) = \omega(a) - b$ for all $(a, b) \in A \times B$.

- Suppose that, for some $a \in A$, $\omega(a)$ is bounded. Then:
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  \text{Comp}^B(\infty, B \setminus \omega(a)) : \text{the unique unbounded restricted component of } B \setminus \omega(a)
  \]
  \[
  \eta^B \omega(a) := B \setminus \text{Comp}^B(\infty, B \setminus \omega(a)) : \text{the } \textit{restricted connected hull of } \omega(a)
  \]

- Suppose that $1 \in H_\omega$. Then:
  \[
  \text{Comp}^B(1, H_\omega) : \text{the restricted component of } 1 \text{ in } H_\omega
  \]
  \[
  \eta^B(A \setminus H_\omega) := A \setminus \text{Comp}^B(1, H_\omega) : \text{the } \textit{restricted connected hull of } A \setminus H_\omega
  \]
Recall:

**Theorem**

Let $A$ be a topological algebra, $B$ a subalgebra of $A$ with unit $e$ and $\omega : A \mapsto 2^B$ a map satisfying $\omega(a - b) = \omega(a) - b$ for all $(a, b) \in A \times B$. Then

$$\partial^B \omega(a) = \{ b \in B : a - b \in \partial^B (A \setminus H_\omega) \}.$$
Now, partially analogous to the previous theorem, we have:
A chain of connected hulls

Now, partially analogous to the previous theorem, we have:

**Theorem**

Let \( A \) be a complex normed algebra with unit \( 1 \) and \( B \) a closed subalgebra of \( A \) such that \( e = 1 \in B \). Let \( \omega : A \to 2^B \) be a mapping such that \( \omega(a - b) = \omega(a) - b \) for all \( (a, b) \in A \times B \), \( \omega(a) \) is bounded and closed in \( B \) for all \( a \in A \) and \( \omega(\lambda 1) = \lambda \omega(1) \) for all \( \lambda \in \mathbb{C} \). If \( 1 \in H_\omega \) then

\[
\eta^B \omega(a) \subseteq \{ b \in B : a - b \in \eta^B (A \setminus H_\omega) \}.
\]
Recall:

\[ \partial_B(a) = \{ \lambda \in \mathbb{C} : a - \lambda 1 \in \partial^B(A\setminus A^{-1}) \} \]
A chain of connected hulls

Recall:

\[ \partial_B(a) = \{ \lambda \in \mathbb{C} : a - \lambda 1 \in \partial^B(A \setminus A^{-1}) \} \]

Now we define:

\[ \eta_B(a) = \{ \lambda \in \mathbb{C} : a - \lambda 1 \in \eta^B(A \setminus A^{-1}) \} \]
Theorem

Let $A$ be a complex Banach algebra with unit $1$ and let $a \in A$. Then

1. $\eta_A(a) = \varepsilon(a)$. 

Corollary

Let $A$ be a complex Banach algebra with unit $1$, $B$ a closed subalgebra of $A$ such that $1 \in B$ and $a \in A$. Then

1. If $a \in C$, then $\varepsilon(a) = \eta_B(a) = \eta_\sigma(a) = \{a\}$.
2. If $a \notin C$, then $\varepsilon(a) \subseteq \eta_B(a) \subseteq C$, where $\eta_B(a)$ is largest when $B = C$, in which case $\eta_B(a) = C$, and smallest when $B = A$, in which case $\eta_B(a) = \varepsilon(a)$.
Theorem

Let $A$ be a complex Banach algebra with unit $1$ and let $a \in A$. Then

1. $\eta_A(a) = \varepsilon(a)$.
2. $\eta\sigma(a) \subseteq \eta\mathcal{C}(a)$.
A chain of connected hulls

Theorem

Let $A$ be a complex Banach algebra with unit 1 and let $a \in A$. Then

1. $\eta_A(a) = \varepsilon(a)$.
2. $\eta\sigma(a) \subseteq \eta\mathbb{C}(a)$.
   - If $a \in \mathbb{C}$, then $\eta\mathbb{C}(a) = \eta\sigma(a)$. 

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Theorem

Let $A$ be a complex Banach algebra with unit $1$ and let $a \in A$. Then

1. $\eta_A(a) = \varepsilon(a)$.
2. $\eta_{\sigma}(a) \subseteq \eta_{\mathbb{C}}(a)$.
   - If $a \in \mathbb{C}$, then $\eta_{\mathbb{C}}(a) = \eta_{\sigma}(a)$.
   - If $a \in A \setminus \mathbb{C}$, then $\eta_{\mathbb{C}}(a) = \mathbb{C}$.
A chain of connected hulls

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Let $A$ be a complex Banach algebra with unit 1 and let $a \in A$. Then

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   - If $a \in \mathbb{C}$, then $\eta_C(a) = \eta\sigma(a)$.
   - If $a \in A \setminus \mathbb{C}$, then $\eta_C(a) = \mathbb{C}$.

Corollary

Let $A$ be a complex Banach algebra with unit 1, $B$ a closed subalgebra of $A$ such that $1 \in B$ and $a \in A$.

1. If $a \in \mathbb{C}$, then $\varepsilon(a) = \eta_B(a) = \eta\sigma(a) = \{a\}$. 

Mouton, Harte  
Boundary and exponential spectra, restricted topology
Theorem

Let $A$ be a complex Banach algebra with unit 1 and let $a \in A$. Then

1. $\eta_A(a) = \varepsilon(a)$.
2. $\eta\sigma(a) \subseteq \eta_C(a)$.
   - If $a \in C$, then $\eta_C(a) = \eta\sigma(a)$.
   - If $a \in A \setminus C$, then $\eta_C(a) = \mathbb{C}$.

Corollary

Let $A$ be a complex Banach algebra with unit 1, $B$ a closed subalgebra of $A$ such that $1 \in B$ and $a \in A$.

1. If $a \in C$, then $\varepsilon(a) = \eta_B(a) = \eta\sigma(a) = \{a\}$.
2. If $a \not\in C$, then $\varepsilon(a) \subseteq \eta_B(a) \subseteq \mathbb{C}$,
A chain of connected hulls

Theorem

Let $A$ be a complex Banach algebra with unit $1$ and let $a \in A$. Then

1. $\eta_A(a) = \varepsilon(a)$.
2. $\eta\sigma(a) \subseteq \eta_C(a)$.
   - If $a \in \mathbb{C}$, then $\eta_C(a) = \eta\sigma(a)$.
   - If $a \in A \setminus \mathbb{C}$, then $\eta_C(a) = \mathbb{C}$.

Corollary

Let $A$ be a complex Banach algebra with unit $1$, $B$ a closed subalgebra of $A$ such that $1 \in B$ and $a \in A$.

1. If $a \in \mathbb{C}$, then $\varepsilon(a) = \eta_B(a) = \eta\sigma(a) = \{a\}$.
2. If $a \notin \mathbb{C}$, then $\varepsilon(a) \subseteq \eta_B(a) \subseteq \mathbb{C}$, where $\eta_B(a)$ is largest when $B = \mathbb{C}$, in which case $\eta_B(a) = \mathbb{C}$.
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2. If $a \not\in \mathbb{C}$, then $\varepsilon(a) \subseteq \eta_B(a) \subseteq \mathbb{C}$, where $\eta_B(a)$ is largest when $B = \mathbb{C}$, in which case $\eta_B(a) = \mathbb{C}$, and smallest when $B = A$, in which case $\eta_B(a) = \varepsilon(a)$.
Final remarks and an application

Let $A$ be a complex Banach algebra with unit $1$, $B$ a closed subalgebra of $A$ such that $1 \in B$ and $a \in A$. If $a \in C$, then $\partial \sigma(a) = \partial C(a) = \partial B(a) = \partial A(a) = S \partial(a) = \sigma(a) = \epsilon(a) = \eta_A(a) = \eta_B(a) = \eta_C(a) = \{a\}$. If $a \not\in C$, then $\partial \sigma(a) \subseteq \partial C(a) \subseteq \partial B(a) \subseteq \partial A(a) = S \partial(a) \subseteq \sigma(a) \subseteq \epsilon(a) = \eta_A(a) \subseteq \eta_B(a) \subseteq \eta_C(a) = C$. 

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1. If $a \in \mathbb{C}$, then

$$
\partial\sigma(a) = \partial\mathbb{C}(a) = \partial_B(a) = \partial_A(a) = S_\theta(a) = \sigma(a) = \varepsilon(a) = \eta_A(a) = \eta_B(a) = \eta\mathbb{C}(a) = \eta\sigma(a) = \{a\}.
$$

2. If $a \not\in \mathbb{C}$, then

$$
\partial\sigma(a) = \partial\mathbb{C}(a) \subseteq \partial_B(a) \subseteq \partial_A(a) = S_\theta(a) \subseteq \sigma(a) \subseteq \varepsilon(a) = \eta_A(a) \subseteq \eta_B(a) \subseteq \eta\mathbb{C}(a) = \mathbb{C}.
$$
Note that there is a type of duality between the boundary spectrum $S_{\partial}(a)$ and the exponential spectrum $\varepsilon(a)$:

$$S_{\partial}(a) = \partial_A(a) \quad \text{and} \quad \varepsilon(a) = \eta_A(a)$$
Final remarks and an application

Note that there is a type of duality between the boundary spectrum $S_{\partial}(a)$ and the exponential spectrum $\varepsilon(a)$:

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The “connected hull” corresponding to $\partial \sigma(a)$ via this duality is $\eta \sigma(a) = \{a\}$ if $a \in \mathbb{C}$.
Note that there is a type of duality between the boundary spectrum $S_\partial(a)$ and the exponential spectrum $\varepsilon(a)$:

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The “connected hull” corresponding to $\partial \sigma(a)$ via this duality is $\eta \sigma(a) = \{a\}$ if $a \in C$ and $C$ if $a \in A \setminus C$. 

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Γ: the circle with centre 0 and radius 1 in \( \mathbb{C} \)

\( D \): the closed disk with centre 0 and radius 1 in \( \mathbb{C} \)

**Example**

Let \( A = C(\Gamma) \), the Banach algebra of complex-valued functions which are continuous on \( \Gamma \), and \( B = A(D) \), the closed subalgebra of \( A \) consisting of all functions which are continuous on \( D \) and analytic on its interior. Let \( a = z : \lambda \mapsto \lambda \) be the identity function on \( D \). Then \( \partial_B(a) = \Gamma \) and \( \eta_B(a) = D \).
Certain results that are known to hold for the boundary spectrum \( S_\partial(a) = \partial_A(a) \) can be generalised by replacing \( A \) by \( B \), with \( B \) a closed subalgebra of \( A \) containing the unit of \( A \).
Certain results that are known to hold for the boundary spectrum $S_{\partial}(a) = \partial_A(a)$ can be generalised by replacing $A$ by $B$, with $B$ a closed subalgebra of $A$ containing the unit of $A$.

**Theorem (Mouton, 2009)**

Let $A$ be a complex Banach algebra with unit $1$. Let $a \in A$ and let $f$ be a complex valued function which is analytic and one-to-one on a neighbourhood of $\sigma_A(a)$. Then $S_{\partial}(f(a)) = f(S_{\partial}(a))$, i.e. $\partial_A(f(a)) = f(\partial_A(a))$. 

**Theorem (Mouton, Harte, 2017)**

Let $A$ be a complex Banach algebra with unit $1$ and let $B$ be a closed subalgebra of $A$ such that $1 \in B$. Let $a \in B$ and let $f$ be a complex valued function which is analytic and one-to-one on a neighbourhood of $\sigma_B(a)$. Then $\partial_B(f(a)) = f(\partial_B(a))$. 

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THANK YOU