Stability of the space of square-summable sequences with respect to convolution

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This is joint work with Thomas J. Ransford.
Throughout this talk, $G$ stands for a unimodular locally compact topological group with left Haar measure $\lambda$.

A weight function on $G$ is a function whose range is in $(0, \infty)$.

Given $G$, $w$ and $p \geq 1$, the weighted $L_p$-space on $G$ with weight $w$ is defined as follows:

$$L_p(G, w) := \left\{ f : G \to \mathbb{C} : \|f\|_{p,w} < \infty \right\},$$

with

$$\|f\|_{p,w} := \|wf\|_p = \left( \int_G w(x)^p |f(x)|^p \, d_L\lambda(x) \right)^{1/p}.$$
Let $f, g : G \to \mathbb{C}$ be two functions. Their *convolution* $f * g : G \to \mathbb{C}$ is the function defined by the formula

$$(f * g)(x) := \int_G f(y) g(y^{-1}x) \, d_L \lambda(y), \quad (x \in G).$$

**Question 1**
For $p \geq 1$, what can we say about the class $\mathcal{W}_p$ of groups $G$ for which there exists a weight function $w$ such that $L_p(G, w)$ is $*$-stable?

**Question 2**
Given $p \geq 1$ and $G \in \mathcal{W}_p$, what can we say about the class $\mathcal{W}_p(G)$ of weight functions on $G$ for which $L_p(G, w)$ is $*$-stable?
Partial answer to question 2

1. \[ \|f \ast g\|_{1,w} \leq \|f\|_{1,w} \|g\|_{1,w}, \]
   \[ \forall f, g \in L_1(G, w) \]
   if and only if \( w : G \to \mathbb{C} \) is submultiplicative, i.e.
   \[ w(xy) \leq w(x)w(y), \quad \forall x, y \in G. \]

2. \[ \|f \ast g\|_{p,w} \leq \|f\|_{p,w} \|g\|_{p,w}, \]
   \[ \forall f, g \in L_p(G, w) \]
   if \( w^{-q} \) is subconvolutive, i.e.
   \[ (w^{-q} \ast w^{-q})(x) \leq w^{-q}(x), \quad \forall x \in G. \]
Definition & notation.

1. The *stability index of* \((G, w)\):

\[
C(G, w) := \sup \left\{ \frac{\|f*g\|_w}{\|f\|_w \|g\|_w} : f, g \in \ell_2(G, w) \right\}.
\]

2. The *subconvolutivity index of* \((G, w)\):

\[
C_2(G, w)^2 := \sup_{x \in G} \frac{(w^{-2} * w^{-2})(x)}{w^{-2}(x)}.
\]

**Theorem.** \(C(G, w) \leq C_2(G, w)\).
Main problem.
Does there exist a discrete abelian group $G$ and a weight function $w$ on $G$ such that

- $\ell_2(G, w)$ is a convolution algebra.
- $w^{-2}$ is not weakly subconvolutive, i.e.

$$C_2(G, w)^2 = \sup_{x \in G} \frac{(w^{-2} * w^{-2})(x)}{w^{-2}(x)} = \sup_{x \in G} \sum_{y \in G} \frac{w^2(x)}{w^2(y) w^2(y^{-1}x)} = \infty.$$
Theorem 1. (Kuznetsova)
Suppose that $\ell_2(G, w)$ is $*$-stable. Then $\sum_{x \in G} \frac{1}{w(x)w(x^{-1})} < \infty$.

Sketch of proof.

- $\ell_2(G, w)$ admits a character $\chi : \ell_2(G, w) \to \mathbb{C}$.
- Riesz’s repr. thm : $\chi(f) = \langle f, h \rangle_w = \sum_{x \in G} w(x)^2 f(x) \overline{h(x)}$.

- Taking $f := \delta_x$ we find that $h(x) = \frac{\chi(\delta_x)}{w(x)^2}$.

- $\sum_{x \in G} \frac{|\chi(\delta_x)|^2}{w(x)^2} = \sum_{x \in G} w(x)^2 |h(x)|^2 = \|h\|^2_w < \infty$.

- We may write $1 = \chi(\delta_0) = \chi(\delta_x \ast \delta_{x^{-1}}) = \chi(\delta_x) \chi(\delta_{x^{-1}})$. Hence by C-S,

$$\sum_{x \in G} \frac{1}{w(x)w(x^{-1})} \leq \left( \sum_{x \in G} \frac{|\chi(\delta_x)|^2}{w(x)^2} \right)^{1/2} \left( \sum_{x \in G} \frac{|\chi(\delta_{x^{-1}})|^2}{w(x^{-1})^2} \right)^{1/2} < \infty.$$
Under certain regularity hypotheses on \( w \), the necessary condition coincide with the sufficient one.

**Theorem 2.**
Assume that there exists a constant \( M > 0 \) such that \( \forall x, y \in G \)

\[
w(xy) \leq M (w(x) + w(y)) \quad \& \quad w(x^{-1}) \leq Mw(x).
\]

Then the following statements are equivalent:

1. \( C_2(G,w) < \infty \);
2. \( C(G,w) < \infty \);
3. \( \sum_{x \in G} \frac{1}{w(x)w(x^{-1})} < \infty \);
4. \( \sum_{x \in G} \frac{1}{w(x)^2} < \infty \).
A new approach.

- Characterize $C(G, w)$ as the norm of a certain operator.
- Exploit this characterization to obtain concrete estimates of $C(G, w)$. 
Notation.

- $H := \ell_2(G)$.
- $H_0 := \text{span}\{e_x : x \in G\}$, where
  
  $$e_x(y) := \begin{cases} 
  1, & y = x; \\
  0, & y \neq x.
  \end{cases}$$

- $\text{Lin}(H_0) := \{ T : H_0 \to H_0 \mid T \text{ linear}\}$.
- $\mathcal{B}(H_0) := \{ T : H_0 \to H_0 \mid T \text{ linear and } \| \cdot \|_H\text{-bounded}\}$.
- $e_y \otimes e_z \in \mathcal{B}(H_0)$ is the rank-1 operator defined by
  
  $$(e_y \otimes e_z)(h) := \langle h, e_z \rangle e_y = h(z)e_y, \quad (h \in H).$$
Consider the map $T_w : H_0 \to \text{Lin}(H_0)$ defined by

$$T_w(h) := \sum_{y \in G} \sum_{z \in G} h(y+z) \frac{w(y+z)}{w(y)w(z)} (e_y \otimes e_z).$$

**Theorem 3.** The following statements are equivalent:

1. $\ell_2(G, w)$ is $\ast$-stable;
2. $T_w$ is a bounded linear map from $H_0$ into $\mathcal{B}(H_0)$.

In this case

$$C(G, w) = \|T_w : H_0 \to \mathcal{B}(H_0)\|.$$
If $T_w$ is a bounded linear map of $H_0$ into $\mathcal{B}(H_0)$, then there is a (unique) bounded linear map $\tilde{T}_w : H \to \mathcal{B}(H)$ that extends $T_w$ in the sense that

$$\tilde{T}_w(h) \upharpoonright H_0 = T_w(h), \quad (h \in H_0).$$

Moreover

$$\|\tilde{T}_w : H \to \mathcal{B}(H)\| = \|T_w : H_0 \to \mathcal{B}(H_0)\|.$$ 

**Theorem 4.** The following statements are equivalent:

1. $C_2(G, w) < \infty$.
2. $\tilde{T}_w$ is a bounded linear map from $H$ into $\mathcal{S}_2(H)$, the class of Hilbert–Schmidt operators on $H$;

In this case

$$C_2(G, w) = \|\tilde{T}_w : H \to \mathcal{S}_2(H)\|.$$
Computing $\|\tilde{T}_w : H \to \mathcal{B}(H)\|$ can be quite complicated.

We seek an estimate for $C(G, w)$ which is smaller than $\|\tilde{T}_w : H \to \mathcal{S}_2(H)\|$, yet simpler to compute than $\|\tilde{T}_w : H \to \mathcal{B}(H)\|$.

One possibility is to consider $\|\tilde{T}_w : H \to \mathcal{S}_p(H)\|$ for the Schatten classes $\mathcal{S}_p(H)$ with $p \in (2, \infty)$. 
Definition.
Given a weighted group \((G, w)\) and \(p \in (2, \infty)\), set
\[
C_p(G, w) := \|\tilde{T}_w : H \to \mathcal{L}_p(H)\|.
\]

Theorem 5.

\[
C_4(G, w)^8 \leq \sup_{s \in G} \sum_{x \in G} \sum_{y \in G} \left( \frac{\sum_{z \in G} w(s)w(x)w(y)w(y^{-1}xs)}{w(z^{-1}x)^2w(z^{-1}y)^2w(z)^2w(y^{-1}zs)^2} \right)^2.
\]
Theorem 6.
For \( n \geq 3 \), define \( w : \mathbb{Z}_n \to (0, \infty) \) by

\[
w(0) = w(1) = \ldots = w(n-2) := 1 \quad \text{and} \quad w(n-1) := t \geq 1.
\]

We have

\[
\frac{C(\mathbb{Z}_n, w)}{C_2(\mathbb{Z}_n, w)} \leq \left( \frac{(n-1) + t^2}{2 + (n-2)t^2} \right)^{1/2}.
\]
**FIGURE:** Graph of $C(\mathbb{Z}_n, w)/C_2(\mathbb{Z}_n, w)$ for various values of $n$. 
THANK YOU for your ATTENTION!