Exactness vs $C^*$-exactness for certain non-discrete groups

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The reduced crossed product

Given a locally compact group $G$, we may equip a $C^*$-algebra $A$ with a continuous action $\alpha: G \to \text{Aut}(A)$ of $G$. There is an associated $C^*$-algebra $A\rtimes_r G$ which encodes $\alpha$, called the reduced crossed product. The symbol "$\rtimes_r G$" may be regarded as a functor from $C^*$-algebras admitting an action of $G$ to ordinary $C^*$-algebras. This functor has stimulated research connecting many areas, such as topological dynamics, coarse geometry, and $C^*$-algebras. In particular, one may ask when $\rtimes_r G$ is an exact functor.
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- In particular, one may ask when $\rtimes_r G$ is an exact functor.
A locally compact group $G$ is said to be exact if for every short exact sequence of $G$-$C^*$-algebras

$$0 \to I \to A \to B \to 0,$$

the associated sequence of reduced crossed products

$$0 \to I \rtimes_r G \to A \rtimes_r G \to B \rtimes_r G \to 0,$$

is also short exact. When $G$ acts trivially on the first sequence, the second sequence becomes

$$0 \to I \otimes C^*_r(G) \to A \otimes C^*_r(G) \to B \otimes C^*_r(G) \to 0.$$

Hence, if $G$ is exact then $C^*_r(G)$ is an exact $C^*$-algebra. In this case we will say $G$ is $C^*$-exact.
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The work of Anantharaman-Delaroche, Kirchberg–Wassermann and Ozawa shows the following.

**Theorem**

Let $G$ be a discrete group. The following are equivalent:

1. $G$ is exact.
2. (Dynamical) $G$ is amenable at infinity.
3. (Metric) $G$ has Yu’s property A.
4. (C$^*$-algebraic) $G$ is C$^*$-exact.
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We have $1 \iff 2 \iff 3 \implies 4$.

**Question:** Does C$^*$-exactness imply exactness?
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**Question:** Does C*-exactness imply exactness?
Recall that $C^*$-exactness and exactness are equivalent for discrete groups. This result was strengthened in 2002 by Anantharaman-Delaroche.

**Theorem**

Let $G$ be a locally compact group with property (W). If $G$ is $C^*$-exact then it is exact. Crann–Tanko showed in 2016 that property (W) is equivalent to inner amenability, providing a large class of groups to which this result applies.
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Cases when the two notions of exactness coincide (cont.)

**Proposition (M.)**

Let $G$ be a locally compact group such that $C^*\text{r}(G)$ admits a tracial state. If $G$ is $C^*$-exact then it is exact. The implication remains true under open unions.

**Theorem (M.)**

Let $G$ be a locally compact group, and $(H_i)_{i \in I}$ a family of open subgroups satisfying the following:

- Each $C^*\text{r}(H_i)$ admits a tracial state.
- The union $\bigcup_{i \in I} H_i$ equals $G$.

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Examples

Proposition (M.)

Let $N$ be an amenable locally compact group, and let $H$ be a discrete group with the property that $\text{ev}$ is the only conjugation invariant state on $\ell_\infty(H)$. If $\alpha: H \to \text{Aut}(N)$ is an action such that there is no $H$-invariant state on $L_\infty(N)$, then $N \rtimes H$ is not inner-amenable.

The reduced $C^*$-algebra $C^*_r(N \rtimes H)$ admits a tracial state since $N$ is an open normal amenable subgroup in $N \rtimes H$ (Kennedy–Raum 2017), hence our results apply.

Forrest–Spronk–Wiersma showed that $R^2 \rtimes F_6$ is such a group, where the action $F_6 \rtimes R^2$ is induced by the inclusion $F_6 \subseteq \text{SL}_2(R)$.

Work in progress: Find more examples of this kind.
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Examples (cont.)

Even among inner amenable groups, our results provide new proofs which don't require the machinery of property (W)/inner amenability: a class of groups considered by Suzuki in the context of non-discrete $\mathbb{C}^*$-simplicity. The class of totally disconnected IN groups.
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- A class of groups considered by Suzuki in the context of non-discrete C*-simplicity.
- The class of totally disconnected IN groups.
Suzuki groups

Definition

Let $G$ be a locally compact group with open subgroups $K_n \trianglelefteq L_n \leq G$ for each $n$. We say $G$ is a Suzuki group if it satisfies the following:

- Each $K_n$ is compact.
- $(K_n)$ forms a neighbourhood base at the identity.
- $\bigcup_n L_n$ is all of $G$.

In 2016, Suzuki provided examples of non-discrete C$^*$-simple groups in this class. It can be shown that each $C^*_r(L_n)$ has a tracial state. Hence the theorem naturally applies to this class.

Note: Forrest–Spronk–Wiersma showed that many groups in this class do not admit a tracial state.
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Totally disconnected IN groups

A locally compact group $G$ is called:

Totally disconnected if its connected components are singletons.

IN if it admits a conjugation invariant compact neighbourhood of the identity.

There are many natural examples of IN groups:

Abelian groups: any compact neighbourhood of the identity.

Discrete groups: the singleton $\{e\}$.

Groups admitting a compact open normal subgroup, e.g., $\prod \Gamma \cdot \Gamma$.

NOT $\text{SL}_n(C)$: elements may be conjugated arbitrarily far from the identity.

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- Abelian groups: any compact neighbourhood of the identity.
- Discrete groups: the singleton $\{e\}$.
- Groups admitting a compact open normal subgroup, e.g., $\prod_{\Gamma} K \rtimes \Gamma$.
- **NOT** $SL_n(\mathbb{C})$: elements may be conjugated arbitrarily far from the identity.

**Proposition**

Let $G$ be a totally disconnected IN group. If $G$ is $C^*$-exact then it is exact.
Why mention these examples?

Theorem (Cave-Zacharias 2018)
If there is a locally compact group which is $C^*$-exact but not exact, there
is necessarily a totally disconnected unimodular such group.

Suzuki groups and totally disconnected IN groups are examples of totally
disconnected unimodular groups, and act as a proof of concept for
techniques tackling this larger class.

Lemma
Let $G$ be a totally disconnected group, and $K \leq G$ a compact open
subgroup. The canonical vector state on the Hecke $C^*$-algebra
$C^*_r(G, K)$ is
tracial if and only if $G$ is unimodular.

Thank you!
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**Theorem (Cave-Zacharias 2018)**

*If there is a locally compact group which is $C^*$-exact but not exact, there is necessarily a totally disconnected unimodular such group.*

**Lemma**

Let $G$ be a totally disconnected group, and $K \leq G$ a compact open subgroup. The canonical vector state on the Hecke $C^*$-algebra $C^r_r(G, K)$ is tracial if and only if $G$ is unimodular.

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Theorem (Cave-Zacharias 2018)

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