The hull-kernel topology on the Berkovich spectrum for commutative Banach rings

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Introduction

Let $A$ denote a commutative unital Banach ring with a complete sub-multiplicative norm $\| \cdot \|$, i.e. the complete metric given by a function $\| \cdot \| : A \to [0, \infty)$ which satisfies the conditions: $\|f\| = 0 \iff f = 0$; $\|f - g\| \leq \|f\| + \|g\|$; $\|fg\| \leq \|f\|\|g\|$ for all $f, g$ in $A$ and $\|1\| = 1$.

From now on, all Banach rings are assumed to be commutative and unital.
Recall that the *Berekovich spectrum* of $A$, write $\mathcal{M}(A)$, is the set of all non-zero multiplicative bounded semi-norms $x$ (also write $\lvert \cdot \rvert_x := x$) on $A$ and is equipped with the pointwise convergence topology (also called the *Gelfand topology*). In here bounded means that $\lvert \cdot \rvert_x \leq \lVert \cdot \rVert$ on $A$.

Notice that if $A$ is a commutative unital complex Banach algebra, then $\mathcal{M}(A)$ is homeomorphic to the usual *Gelfand spectrum* of $A$ due to the Gelfand-Mazur Theorem.

**Example:**
If the set of all integers $\mathbb{Z}$ is endowed with the usual absolute $\lvert \cdot \rvert_\infty$, then the Ostrowski’s Theorem tells us that $\mathcal{M}(\mathbb{Z}) = \{\lvert \cdot \rvert_\infty^a : 0 < a \leq 1\} \cup \{\lvert \cdot \rvert_p^a : a = -\log \varepsilon / \log p : 0 < \varepsilon < 1\} \cup \{\lvert \cdot \rvert_0, \lvert \cdot \rvert_{p,0}\}$, here $\lvert \cdot \rvert_0$ denotes the trivial absolute value, i.e., $\lvert m \rvert_0 \equiv 1$ for $m \neq 0$, and $\lvert \cdot \rvert_{p,0}$ is the seminorm induced by the trivial norm on $\mathbb{Z}/p\mathbb{Z}$. 
Introduction

Theorem

$\mathcal{M}(A)$ is always a non-empty compact Hausdorff space.

(See: V.G. Berkovich, Spectral theory and analytic geometry over non-Archimedean fields, Mathematical surveys and monographs, AMS, (1990).)
Regularity for Banach rings

Recall that a complex unital commutative Banach algebra $B$ is said to be regular if every $w^*$-closed subset $E$ of the character space $\Delta(B)$ and $\varphi \notin E$, there exists an element $f \in B$ such that $\hat{f}(E) \equiv 0$ and $\hat{f}(\varphi) \neq 0$, where $\hat{f}$ denotes the usual Gelfand transform of $f$.

Example:
Urysohn’s Lemma implies that $C(X)$ is regular for any compact Hausdorff space $X$.

Fact: $B$ is regular if and only if $\Delta(B)$ is Hausdorff in the hull-kernel topology.
Regularity for Banach rings

For each element $x$ in $\mathcal{M}(A)$, put $\ker x := \{ f \in A : |f|_x = 0 \}$ the kernel of $x$. Note that the $\ker x$ is a norm closed prime ideal of $A$.

Let $\mathcal{H}(x)$ be the completion of the quotient field of $A/\ker x$ with respect to the absolute value, also write $|\cdot|_x$, induced by the semi-norm $|\cdot|_x$.

An absolute value $|\cdot|$ on a field $K$ in here means that it is a multiplicative norm.

For convenience we write $\prod^\infty(A)$ for the set of all elements $u$ in

$$\prod_{x \in \mathcal{M}(A)} \mathcal{H}(x)$$

which are bounded, i.e.,

$$\|u\|_\infty := \sup_{x \in \mathcal{M}(A)} |u(x)|_{\mathcal{H}(x)} < \infty.$$ 

Also, $\prod^\infty(A)$ is equipped with the $\|\cdot\|_\infty$-topology. Then $\prod^\infty(A)$ becomes a commutative unital Banach ring.
Regularity for Banach rings

As in the case of a complex Banach algebra, the Gelfand transform $\mathcal{G}$ is defined by

$$\mathcal{G} : A \longrightarrow \Pi^\infty(A) : f \mapsto \hat{f}$$

where $\hat{f}(x) := f + \ker x \in \mathcal{H}(x)(:= \text{Frac}(A/ \ker x))$.

**Definition**

A commutative unital Banach ring $A$ is said to be *regular* if the condition hold:

if $E$ is a Gelfand closed subset of $\mathcal{M}(A)$ and $x_0 \in \mathcal{M}(A) \setminus E$, then there exists an element $f \in A$ such that $\hat{f}(x_0) \neq 0$ and $\hat{f}(E) \equiv 0$.

It is clear that if there is $x \in \mathcal{M}(A)$ with $\ker x = 0$, then $A$ must not be regular. For example, the ring of integers $\mathbb{Z}$ is not regular.
Now the \textit{hull-kernel topology} on $\mathcal{M}(A)$ is given by the following way. For a subset $E$ of $\mathcal{M}(A)$, the \textit{hull-kernel closure} of $E$, write $\overline{E}^{hk}$, is defined by

$$\overline{E}^{hk} := \{x \in \mathcal{M}(A) : \bigcap_{y \in E} \ker y \subseteq \ker x\}.$$ 

One can directly check that the hull-kernel topology on $\mathcal{M}(A)$ is a $T_0$ topology. Also, the Gelfand topology is always stronger than the hull-kernel topology on $\mathcal{M}(A)$. 
Regularity for Banach rings

Proposition

Let $A$ be a unital commutative Banach ring. Then the following statements are equivalent.

(i) $A$ is regular.

(ii) The Gelfand topology and the hull-kernel topology are equivalent on $\mathcal{M}(A)$.

(iii) $\mathcal{M}(A)$ is Hausdorff in the hull-kernel topology.
**Regularity for Banach rings**

**Question:**

If $k$ is a complete valuation field and $X$ is a topological space, is the Banach algebra of all bounded continuous $k$-valued functions defined on $X$, write $C^b(X, k)$, regular?
Recall that a topological space $T$ is said to be zero-dimension if the set of all closed and open (clopen) subsets of $T$ forms an open basis of $T$. Let $T$ be a zero-dimension space. Put

$$CO(T) := \{ U : U \text{ is a clopen subset of } T \}.$$  

Recall that an ultrafilter of $CO(T)$ is a collection $\varphi$ of clopen subsets of $T$ which satisfies the following conditions: (i) the empty set $\emptyset \not\in \varphi$; (ii) if $A, B \in \varphi$ implies $A \cap B \in \varphi$; (iii) if $A \subseteq B$ with $A \in \varphi$ and $B \in CO(T)$, then $B \in \varphi$; and (iv) $A \in \varphi$ or the complement $A^c \in \varphi$ for each $A \in CO(T)$. 

Regularity for Banach rings

The ultrafilter space, write $UF(T)$, of $T$ is the set of all ultrafilters of $CO(T)$ which is equipped with the topology given by the following way:

A subset $\mathcal{W} \subseteq UF(T)$ is open if for each $\varphi_0 \in \mathcal{W}$, there exists an element $J_0 \in \varphi_0$ such that $\varphi \in \mathcal{W}$ whenever $J_0 \in \varphi$. 
Regularity for Banach spaces

Let $C^b(T, k)$ be the space of bounded $k$-valued continuous functions defined on $T$ and let $|\cdot|_k$ be the valuation of $k$. For each $\varphi \in UF(T)$ and $f \in C^b(T, k)$, put

$$|f|_\varphi := \inf_{J \in \varphi} \sup_{t \in J} |f(t)|_k.$$  \hfill (1)

The following result was first shown by Berkovich for the case of $T$ being discrete.

**Theorem**

*(Mihara): the Berkovich spectrum of $C^b(T, k)$ can be identified with the ultrafilter space $UF(T)$ under the homeomorphism*

$$\Psi : \varphi \in UF(T) \mapsto |\cdot|_\varphi \in \mathcal{M}(C^b(T, k))$$

Regularity for Banach rings

Theorem
Using the notation as above, if $T$ is a zero-dimension space, then the commutative unital Banach ring $C^b(T, k)$ is regular. Consequently, the Berkovich spectrum $\mathcal{M}(C^b(T, k))$ is Hausdorff in the hull-kernel topology.
Regularity for Banach rings

Proof.
Let $E$ be a closed subset of $\mathcal{M}(C^b(T, k))$ and $\varphi_0 \notin E$. Now for each element $\varphi_l \in E$, then by the maximality of $\varphi_l$ and $\varphi_0$, there exists a clopen subset $J_l$ of $T$ such that $J_l \in \varphi_l$ but $J_l \notin \varphi_0$.
Define a $k$-valued function $g_l$ on $T$ by $g_l := 1 - \chi_{J_l}$. Then $g_l \in C^b(T, k)$ because the set $J_l$ is clopen.
Now put $V_l := \{\varphi \in UF(T) : J_l \in \varphi\}$. Then by the definition of the topology on $UF(T)$, $V_l$ is an open neighborhood of $\varphi_l$ in $UF(T)$ and thus, $V_l$ is also open in $\mathcal{M}(C^b(T, k))$ under the identification $\Psi$ by using Mihara’s Theorem above. Also Eq 1 gives $|g_l|_{\varphi} = |g_l|_{\varphi_l} = 0$ for all $\varphi \in V_l$ and $|g_l|_{\varphi_0} = 1$ because $J_l^c \in \varphi_0$.
On the other hand, since $E$ is compact, there are finitely many elements, say $\varphi_1, \ldots, \varphi_N$, in $E$ such that $E \subseteq V_1 \cup \cdots \cup V_N$, where $V_j$’s are the corresponding open neighborhood of $\varphi_l$ in $\mathcal{M}(C^b(T, k))$ constructed as above.
Let $f$ be the product $g_1 \cdots g_N \in C^b(T, k)$. Then $|f|_{\varphi_0} = 1$ and $\hat{f}(E) \equiv 0$. Thus, $C^b(T, k)$ is regular and the last assertion follows from Proposition 3 immediately. \qed
Using the same argument as the Theorem above, we also have the following result.

**Proposition**

Let $I$ be a discrete set. Let $\{ k_i : i \in I \}$ be a family of complete valuation fields. Then the Banach ring $\prod_{i \in I}^{b} k_i := \{ a \in \prod_{i \in I} k_i : \sup_{i} |a(i)|_{k_i} < \infty \}$ is regular.
Finiteness of $\mathcal{M}(A)$

By considering the dual transform of $G$, that is,

$$\hat{G} : \mathcal{M}(\Pi^\infty(A)) \to \mathcal{M}(A)$$

given by $\hat{G}(\varphi)(f) := |\hat{f}|_\varphi$ for $\varphi \in \mathcal{M}(\Pi^\infty(A))$ and $f \in A$, we have the following

**Proposition**

*If the image of Gelfand transform $G$ is dense in $\Pi^\infty(A)$, then the spectrum $\mathcal{M}(A)$ is finite.*

**Question**

(i) *Does the converse of the above Proposition hold?*

(ii) *Is $A$ is regular if $\mathcal{M}(A)$ is finite?*
Finiteness of $\mathcal{M}(A)$

Recall that the spectral radius $\rho(f)$ of an element $f$ in $A$ is defined by $\rho(f) := \max_{x \in \mathcal{M}(A)} |f|_x$.

$$\rho(f) = \lim_n \|f^n\|^{1/n}.$$ 

From this when $A$ is uniform, i.e. $\|f^2\| = \|f\|^2$ for all $f$ in $A$, then the Gelfand transform $\mathcal{G} : A \to \Pi^\infty(A)$ is an isometry.
Finiteness of $\mathcal{M}(A)$

Now if $A^u$ denotes the completion of $A/\ker \rho$ under the spectral radius $\rho$, then $A^u$ is a uniform Banach ring.

Notice that from the definition of spectral radius $\rho$, one can directly check that the map $\tilde{\phi}_u : \mathcal{M}(A^u) \rightarrow \mathcal{M}(A)$ which is induced by the canonical ring homomorphism $\phi_u : A \rightarrow A^u$ is a homeomorphism.
Finiteness of $\mathcal{M}(A)$

Theorem

Assume that $x$ is non-trivial on $\mathcal{H}(x)$ for all $x \in \mathcal{M}(A)$. Then the Berkovich spectrum $\mathcal{M}(A)$ is finite if and only if the image $\mathcal{G}(A)$ is dense in $\Pi^\infty(A)$. 
Finiteness of $\mathcal{M}(A)$

Outline Proof ($\Rightarrow$):
Let $\mathcal{M}(A) = \{x_1, ..., x_n\}$.

- If $G(A^u)$ is dense in $\Pi^\infty(A^u)$, then $G(A)$ is also dense in $\Pi^\infty(A)$.
  Hence, we may assume that $A$ is uniform.

- Assume that $\mathcal{M}(A)$ is finite and each element $x$ in $\mathcal{M}(A)$ is non-trivial on $\mathcal{H}(x)$. Then $\ker x$ is a maximal ideal for all $x \in \mathcal{M}(A)$.

- We need the following Artin-Whaples approximation theorem: if $x_1, ..., x_n$ are the pairwise inequivalent non-trivial absolute values on a field $K$, then for any $\varepsilon > 0$ and $a_1, ..., a_n$ in $K$, one can find an element $a$ in $K$ such that $|a - a_k|_{x_k} < \varepsilon$ for all $k = 1, ..., n$.

Then one can show that $G(A) = A/\ker x_1 \times \cdots \times A/\ker x_n = \mathcal{H}(x_1) \times \cdots \times \mathcal{H}(x_n)$ as desired.
Corollary

If $A$ is uniform with finite spectrum, then $A$ is regular.

Proof.

By Theorem 9, we see that the Gelfand transform of $A$ is an isometric isomorphic from $A$ onto $\Pi^\infty(A) := \prod_{x \in \mathcal{M}(A)}^b \mathcal{H}(x)$. So the result follows from Proposition 6 immediately. \qed
Finiteness of $\mathcal{M}(A)$

Remark
When $A$ is a commutative unital Banach algebra over a valuation field, then the above theorem can be directly obtained by the Shilov Idempotent Theorem: for each non-empty closed and open subset $D$ of $\mathcal{M}(A)$, there is an idempotent $e$ in $A$ such that $\hat{e} \equiv 1$ on $D$ and is equal to 0 outside $D$.

Indeed, as in the proof of the Theorem above, we may assume that $A$ is uniform. Let $\mathcal{M}(A) = \{x_1, ..., x_n\}$ as before. To apply Shilov Idempotent Theorem, we can obtain the idempotents $e_1, ..., e_n$ in $A$ such that $\hat{e_i}(x_j) = \delta_{ij}1 \in \mathcal{H}(x_j)$. Hence, we have $e_ie_j = \delta_{ij}1$ because the Gelfand transform $G$ is a ring monomorphism. Then $A = e_1A \times \cdots \times e_nA$ as the rings isomorphic. From this, we can see that the completion of $A/\ker x_k$ with respect to $\bar{x}_k$ is a valuation field, where $\bar{x}_k$ is the multiplicative norm on $A/\ker x_k$ induced by $x_k$. This, together with the isometric property of $G$, the canonical map (the restriction of $G$) is an isometric isomorphism from $e_kA$ onto $A/\ker x_k = A/\ker x_k |_{\bar{x}_k} = \mathcal{H}(x_k)$. This is as desired.
Thank you!!