Closed ideals of the algebra of bounded operators on a Banach space

Niels Laustsen

Lancaster University, UK

Banach Algebras and Applications

University of Manitoba, Winnipeg

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Let $X$ be a Banach space over the scalar field $K = \mathbb{R}$ or $K = \mathbb{C}$, and consider the Banach algebra

$$B(X) = \{ T : X \to X : T \text{ bounded and linear} \},$$

‘an operator’

**Main question:** what are the closed (two-sided) ideals of $B(X)$?

**Example.** If $\dim X < \infty$, then $B(X) \cong M_n$, where $n = \dim X$, and $M_n$ is simple: $\{0\}$ and $M_n$ are the only ideals.

**More general fact.** Let $X$ be a non-zero Banach space. Then the closure of

$$\mathcal{F}(X) := \{ T \in B(X) : \dim T(X) < \infty \}$$

is a non-zero closed ideal of $B(X)$, and it is the smallest such.

If $X$ is ‘nice’, then $\overline{\mathcal{F}(X)} = \mathcal{K}(X)$, the ideal of compact operators.
Classical ideal classification results

**Theorem** (Calkin, *Ann. of Math.* 1941). Let $H$ be a separable, $\infty$-dimensional Hilbert space. Then there are exactly three closed ideals in $\mathcal{B}(H)$, namely

$$\{0\} \subset \mathcal{K}(H) \subset \mathcal{B}(H).$$

**Theorem** (Gohberg–Markus–Feldman, *Bul. Akad. Ştiinţe RSS Moldoven* 1960). Let $X = c_0$ or $X = \ell_p$, where $1 \leq p < \infty$. Then there are exactly three closed ideals in $\mathcal{B}(X)$, namely

$$\{0\} \subset \mathcal{K}(X) \subset \mathcal{B}(X).$$


Daws (*Math. Proc. Cambridge Phil. Soc.* 2006) generalized the above to $c_0(\Gamma)$ and $\ell_p(\Gamma)$ for an arbitrary uncountable index set $\Gamma$. 

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For $n \in \mathbb{N}$, let $\ell_2^n = \mathbb{K}^n$ with the Euclidean norm.


$$X = (\ell_2^1 \oplus \ell_2^2 \oplus \cdots \oplus \ell_2^n \oplus \cdots)_{c_0}$$

$$= \{(x_n)_{n=1}^\infty : x_n \in \ell_2^n \ (n \in \mathbb{N}) \text{ and } \|x_n\| \to 0\}.$$

There are exactly four closed ideals in $\mathcal{B}(X)$, namely

$$\{0\} \subset \mathcal{K}(X) \subset \mathcal{G}_{c_0}(X) \subset \mathcal{B}(X),$$

where

$$\mathcal{G}_{c_0}(X) = \{X \xrightarrow{S} c_0 \xrightarrow{T} X : S \in \mathcal{B}(X, c_0) \text{ and } T \in \mathcal{B}(c_0, X)\}.$$


$$X^* = (\ell_2^1 \oplus \ell_2^2 \oplus \cdots \oplus \ell_2^n \oplus \cdots)_{\ell_1}$$

$$= \{(x_n)_{n=1}^\infty : x_n \in \ell_2^n \ (n \in \mathbb{N}) \text{ and } \sum_{n=1}^\infty \|x_n\| < \infty\}.$$

There are exactly four closed ideals in $\mathcal{B}(X^*)$, namely

$$\{0\} \subset \mathcal{K}(X^*) \subset \mathcal{G}_{\ell_1}(X^*) \subset \mathcal{B}(X^*).$$
‘Exotic’ spaces: the breakthrough of Argyros–Haydon

**Theorem** (Argyros–Haydon, *Acta Math.* 2011). There is a Banach space $X_{AH}$ which has ‘very few’ operators:

$$\mathcal{B}(X_{AH}) = \mathbb{K} I + \mathcal{K}(X_{AH}),$$

where $I$ is the identity operator.

Moreover, $X_{AH}$ has a Schauder basis and $X_{AH}^* \cong \ell_1$.

Consequence: $\mathcal{B}(X_{AH})$ contains exactly three closed ideals, namely

$$\{0\} \subset \mathcal{K}(X_{AH}) \subset \mathcal{B}(X_{AH}).$$

This work was the culmination of a long sequence of constructions of ‘exotic’ Banach spaces initiated by Tsirelson (*Funkcional. Anal. i Priložen* 1972).

Other key contributions:

- Bourgain and Delbaen (*Acta Math.* 1980);
- Schlumprecht (*Israel J. Math.* 1991);
Ideal classifications for modifications of the Argyros–Haydon space

- Tarbard (J. London Math. Soc. 2012): examples where $\mathcal{B}(X)$ has precisely $n$ closed ideals, linearly ordered, for each integer $n \geq 4$;
- Motakis–Puglisi–Zisimopoulou (Indiana Univ. Math. J. 2016): for each countable, compact metric space $\Omega$, there is a Banach space $X$ such that
  $$\mathcal{B}(X)/\mathcal{K}(X) \cong C(\Omega).$$
- Kania–L (Indiana Univ. Math. J. 2017): $X_{AH}$ contains a closed subspace $Y$ of infinite codimension such that every operator $Y \to X_{AH}$ has the form ‘scalar times inclusion map plus compact’. Hence $\mathcal{B}(X_{AH} \oplus Y)$ has exactly 6 closed ideals:

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\begin{tikzcd}
\mathcal{B}(X_{AH} \oplus Y) \\
\rightarrow \\
M_1 \\
\rightarrow \\
M_1 \cap M_2 \\
\rightarrow \\
\mathcal{K}(X_{AH} \oplus Y) \\
\rightarrow \\
\{0\}.
\end{tikzcd}
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Assuming CH, Koszmider (Proc. Amer. Math. Soc. 2005) constructed a compact Hausdorff space $\Omega$ such that

- $\mathcal{B}(C(\Omega)) = \{\alpha I + S : \alpha \in \mathbb{K}, S \text{ has separable range}\},$
- each separable subspace of $C(\Omega)$ has a superspace which is complemented in $C(\Omega)$ and isomorphic to $c_0$.

Kania and Kochanek (J. Op. Th. 2014) used this to classify the closed ideals of $\mathcal{B}(C(\Omega))$:

$$\{0\} \subset \mathcal{K}(C(\Omega)) \subset \mathcal{G}_{c_0}(C(\Omega)) \subset \mathcal{B}(C(\Omega)),$$

where

$$\mathcal{G}_{c_0}(C(\Omega)) =$$

$$\{ C(\Omega) \xrightarrow{S} c_0 \xrightarrow{T} C(\Omega) : S \in \mathcal{B}(C(\Omega), c_0) \text{ and } T \in \mathcal{B}(c_0, C(\Omega)) \}.$$  

This result has also been obtained independently by Brooker (unpublished).

Work in progress (Koszmider–L): construct such a space $\Omega$ within ZFC.
Return to classical spaces

After the classification results with Loy–Read and Schlumprecht–Zsák, I expected many similar results. However, no other ‘classical’ Banach space has had its closed operator ideals classified since. It appears a difficult problem!

Evidence: many classical Banach spaces are known to have very large and/or intricate lattices of closed operator ideals.

- Berkson and Porta (1969) studied the closed ideals of \( \mathcal{B}(L_p[0,1]) \) for \( p \in (1, 2) \cup (2, \infty) \); they are not linearly ordered.
  Pietsch: there are infinitely many.
- Porta (Bull. Amer. Math. Soc. 1969): let \( P \subseteq [1, 2) \cup (2, \infty) \) be countably infinite, and set
  \[
  X = \left( \bigoplus_{p \in P} \ell_p \right) \ell_2.
  \]

Then there is an injection

\[
[N]^{<\infty} \rightarrow \text{ideal } \mathcal{B}(X)
\]

which preserves inclusions in both directions.

Note: we may arrange that \( X \) is reflexive and isometric to \( X^* \). This appears to be the first example of a separable Banach space \( X \) such that \( \mathcal{B}(X) \) has infinitely many closed ideals.
Further examples of classical spaces with many closed operator ideals

- Porta (Studia Math. 1970/71) also initiated the study of the closed ideals of $\mathcal{B}(\ell_p \oplus \ell_q)$ for $1 \leq p < q < \infty$; continued by Volkmann (ibid. 1976). Notably: $\mathcal{B}(\ell_p \oplus \ell_q)$ has precisely two maximal ideals. Pietsch: does $\mathcal{B}(\ell_p \oplus \ell_q)$ contain infinitely many closed ideals? Schlumprecht and Zsák (J. Reine Angew. Math. 2018): yes — even continuum many! (Also for $\ell_p \oplus c_0$; some cases by Sirotkin–Wallis and Freeman–Schlumprecht–Zsák).

- Johnson, Pisier and Schectman (preprint 2018): $\mathcal{B}(X)$ contains continuum many closed ideals for
  \[ X = L_1[0, 1], \quad X = C[0, 1] \quad \text{and} \quad X = \ell_\infty \ (\cong L_\infty[0, 1]). \]

- L–Loy (Banach Center Publ. 2005): for Figiel’s non-Cartesian reflexive Banach space
  \[ F = \left( \bigoplus_{k=1}^{\infty} \ell_{p_k}^{n_k} \right) \ell_q, \]
  where $p_1 > p_2 > \cdots > 2$, $\inf p_k \geq q > 1$ and the sequence $(n_k)$ increases sufficiently rapidly, there is an injection
  \[ \text{ideal } \mathcal{P}(\mathbb{N})/\mathbb{N}^{<\infty} \longrightarrow \overline{\text{ideal } \mathcal{B}(F)} \]
  which preserves the order in both directions.
New results

Joint work with:
- Kevin Beanland (Washington and Lee University, VA, USA) and
- Tomasz Kania (Czech Academy of Sciences).

Set-up: Let $X$ be a Banach space with an unconditional basis $(b_j)_{j \in \mathbb{N}}$. Thus, for each $x \in X$, there is a unique scalar sequence $(\alpha_j)$ such that

$$x = \sum_{j=1}^{\infty} \alpha_j b_j,$$

where the series converges unconditionally, that is, the series

$$\sum_{j \in M} \alpha_j b_j$$

converges for each $M \subseteq \mathbb{N}$. Call the sum $P_M x$; this defines an idempotent operator $P_M \in \mathcal{B}(X)$.

Definition. A closed ideal $\mathcal{I}$ of $\mathcal{B}(X)$ is spatial if

$$\mathcal{I} = \langle P_M \rangle$$

for some non-empty $M \subseteq \mathbb{N}$. A spatial ideal $\mathcal{I}$ is non-trivial if

$$\mathcal{K}(X) \subsetneq \mathcal{I} \subsetneq \mathcal{B}(X).$$
Let $X$ denote either Tsirelson’s space $T$ or the Schreier space $X[S_n]$ of order $n$ for some $n \in \mathbb{N}$.

(i) The family of non-trivial spatial ideals of $\mathcal{B}(X)$ is non-empty and has no minimal or maximal elements.

(ii) Let $\mathcal{I} \subsetneq \mathcal{J}$ be spatial ideals of $\mathcal{B}(X)$. There is a family $\{\Gamma_L : L \in \Delta\}$ such that:
   - the index set $\Delta$ has the cardinality of the continuum;
   - for each $L \in \Delta$, $\Gamma_L$ is an uncountable chain of spatial ideals of $\mathcal{B}(X)$ such that
     \[ \mathcal{I} \subsetneq \mathcal{L} \subsetneq \mathcal{J} \quad (\mathcal{L} \in \Gamma_L), \]
     and $\bigcup \Gamma_L$ is a closed ideal that is not spatial;
   - $\mathcal{L} + \mathcal{M} = \mathcal{J}$ whenever $\mathcal{L} \in \Gamma_L$ and $\mathcal{M} \in \Gamma_M$ for distinct $L, M \in \Delta$.

(iii) The Banach algebra $\mathcal{B}(X)$ contains at least continuum many maximal ideals.
The original Schreier space

**Definition.** A subset $E$ of $\mathbb{N}$ is *admissible* if

$$E \neq \emptyset \quad \text{and} \quad |E| \leq \min E$$

Note: $E$ is finite.

Let $S_1$ be the collection of admissible sets. For $x = (\alpha_j) \in c_{00}$, define

$$\|x\| = \sup \left\{ \sum_{j \in E} |\alpha_j| : E \in S_1 \right\}.$$

The *Schreier space* $X[S_1]$ is the completion of $c_{00}$ with respect to this norm.

**Historical origin.** Banach and Saks (Studia Math. 1930) proved that for $p \in (1, \infty)$, each weakly convergent sequence in $L_p[0,1]$ has a subsequence for which the sequence of arithmetic means is norm-convergent. They asked whether this is also true in $C[0,1]$. Schreier (ibid.) used the above space to produce a counterexample.

\[ S_0 = \{ \{ k \} : k \in \mathbb{N} \}, \]

and for \( n \in \mathbb{N}_0 \), recursively define the \((n + 1)^{\text{st}}\) Schreier family by

\[ S_{n+1} = \left\{ \bigcup_{i=1}^{k} E_i : k \in \mathbb{N}, E_1, \ldots, E_k \in S_n, k \leq \min E_1, E_1 < E_2 < \cdots < E_k \right\}, \]

where \( E_j < E_{j+1} \) means that \( \max E_j < \min E_{j+1} \).

The **Schreier space** \( X[S_n] \) **of order** \( n \) is the completion of \( c_{00} \) wrt the norm

\[ \|x\| = \sup \left\{ \sum_{j \in E} |\alpha_j| : E \in S_n \right\} \quad (x = (\alpha_j) \in c_{00}). \]

*Note:* \( X[S_0] = c_0; S_1 \) is the collection of admissible sets as on previous slide.

*Technical remark:* usually the empty set is also included in \( S_n \).

**Fact.** For each \( n \in \mathbb{N}_0 \), the unit vector basis \( e_j = (0, 0, \ldots, 0, 1_{\text{pos. } j}, 0, \ldots) \) is an unconditional basis for \( X[S_n] \).
Tsirelson’s space: the definition

For $E \subseteq \mathbb{N}$, let $P_E : c_{00} \to c_{00}$ be the corresponding projection, so that

$$P_E \left( \sum_{j \in \mathbb{N}} \alpha_j e_j \right) = \sum_{j \in E} \alpha_j e_j.$$

Recursively define an increasing sequence $(\| \cdot \|_m)_{m \in \mathbb{N}_0}$ of norms on $c_{00}$ by

\[
\|x\|_0 = \max\{ |\alpha_j| : j \in \mathbb{N} \}, \\
\|x\|_{m+1} = \max\{ \|x\|_m \} \cup \\
\left\{ \frac{1}{2} \sum_{j=1}^{k} \|P_{E_j}x\|_m : k \in \mathbb{N}, k \leq \min E_1, E_1 < E_2 < \cdots < E_k \right\},
\]

where $x = (\alpha_j) \in c_{00}$.

**Note:** $\|x\|_m \leq \|x\|_{m+1} \leq \|x\|_{\ell_1} < \infty$, so

\[
\lim_{m \to \infty} \|x\|_m = \sup\{ \|x\|_m : m \in \mathbb{N}_0 \} =: \|x\|_T
\]

exists and defines a norm on $c_{00}$. The completion is Tsirelson’s space $T$.

**Remark.** The above actually defines the dual of Tsirelson’s original space. This approach is due to Figiel and Johnson (*Compos. Math. 1974*).
Theorem (Tsirelson 1972).

- The unit vector basis is an unconditional basis for $T$.
- No Banach space with a subsymmetric basis embeds into $T$. In particular, $T$ contains no copies of $c_0$ or $\ell_p$ for $1 \leq p < \infty$.
- $T$ is reflexive.

Definition. A subsymmetric basis for a Banach space $X$ is an unconditional basis $(b_j)_{j\in\mathbb{N}}$ which is equivalent to each of its subsequences; that is, for each strictly increasing sequence $(m_j)$ in $\mathbb{N}$, the map

$$b_j \mapsto b_{m_j} \quad (j \in \mathbb{N})$$

extends to a bounded, linear isomorphism from $X$ onto $\overline{\text{span}} \{b_{m_j} : j \in \mathbb{N}\}$. 
Proof for Tsirelson’s space: the key tool

**Definition.** Let \( N = \{n_1 < n_2 < \cdots \} \subseteq \mathbb{N} \) be infinite, and let \( J \subseteq \mathbb{N} \) be finite and non-empty. Then \( \sigma(N, J) \) denotes the norm of the formal identity operator from \( \text{span}\{e_{n_j} : j \in J\} \subseteq T \) to \( \ell_1(J) \):

\[
\sigma(N, J) := \sup \left\{ \sum_{j \in J} \alpha_j : \alpha_j \in [0, 1], \left\| \sum_{j \in J} \alpha_j e_{n_j} \right\|_T \leq 1 \right\}.
\]

**Proposition.** Let \( M \subseteq \mathbb{N} \) be infinite subsets of \( \mathbb{N} \). Then TFAE:

1. \( P_N \in \langle P_M \rangle \);
2. \( \langle P_M \rangle = \langle P_N \rangle \);
3. \( P_N(T) \) is isomorphic to \( P_M(T) \);
4. the basic sequence \((e_j)_{j \in M}\) is equivalent to \((e_j)_{j \in \mathbb{N}}\);
5. there is a constant \( C \geq 1 \) such that \( \sigma(N, J) \leq C \) for each ‘interval’ \( J \) in \( \mathbb{N} \) with \( J \cap M = \emptyset \).

**Note:** (d) \( \iff \) (e) is due to Casazza–Johnson–Tzafriri (Israel J. Math. 1984).