FIXED POINT SET FOR SEMIGROUP OF MAPPINGS
ON BANACH SPACES RELATED TO HARMONIC ANALYSIS

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Outline of Talk

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1. The general problem

Let \( K \) be a non-empty compact convex subset of a separated locally convex space and \( S = \{T_s : s \in S\} \) be a semigroup of continuous mappings from \( K \) into \( K \).

**Problem 1.** When is the fixed point set of \( S \)

\[
F(S) = \{x \in K; T_s x = x \text{ for all } x \in S\}
\]

non-empty?

**Problem 2.** Suppose we know that \( F(S) \) is non-empty, what can we say about elements in \( F(S) \)? or span \( F(S) \)?
• **Schauder’s Fixed Point Theorem:** If $S$ is the free commutative semigroup on one generator, then $F(S) \neq \emptyset$.

• **Markov-Kakutani Fixed Point Theorem:** If $S$ is commutative and affine, then $F(S) \neq \emptyset$.

• W.M. Boyce (1969, TAMS). There are two continuous functions $f$ and $g : [0, 1] \to [0, 1]$ which commute under composition, and do not have a common fixed point.
2. Left amenable semigroups

A semigroup $S$ is left amenable if $\ell^\infty(S) =$ space of bounded complex-valued of $S$ has a left invariant mean $m$ i.e. $m \in \ell^\infty(S)^*$, $\|m\| = m(1) = 1$ and $m(\ell_a f) = m(f)$ for all $a \in S$, and $f \in \ell^\infty(S)$.

For a convex subset $C$ of a vector space, a map $\Lambda : C \to C$ is said to be affine if

$$\Lambda(tx + (1 - t)y) = t\Lambda(x) + (1 - t)\Lambda(y) (x, y \in C, t \in [0, 1]).$$

**Theorem** (Day, 1961). A semigroup $S$ is left amenable if and only if $S$ has the following fixed point property:

$(\ast)$ : Whenever $S = \{T_s; s \in S\}$ is a representation of $S$ as continuous affine mappings on a non-empty compact convex subset $K$ of a separated locally convex space, then there is a $x_0 \in K$ such that $T_s(x_0) = x_0$ for all $s \in S$. 


Corollary. Any commutative semigroup is left amenable.

A (discrete) semigroup group $S$ is left reversible if $aS \cap bS \neq \emptyset$ for any $a, b, \in S$.

Examples:

- $S$ commutative or left amenable $\implies S$ is left reversible.
- Any group is left reversible.
- A finite semigroup is left amenable $\iff S$ is left reversible.
- The solvable group: $G =$ all $2 \times 2$ matrices $\begin{bmatrix} x & y \\ 0 & 1 \end{bmatrix}$, $x, y \in \mathbb{R}$, $x \neq 0$ with matrix multiplication (called “$ax + b$” group), is left amenable.
- Free semigroup, or free group on 2 generators is not left amenable.
\[ K = \{ m \in \ell^\infty(S)^*: \|m\| = m(1) = 1 \}, \text{ the set of means on } \ell^\infty(S). \]

\[ S = \{ \ell^*_s; s \in S \}, \quad \langle \ell^*_s m, f \rangle = \langle m, \ell_s f \rangle, \ s \in S, \ m \in K. \]

\[ F(S) = \text{ set of left invariant means on } \ell^\infty(S) \]

Then \( S \) is left amenable \( \iff F(S) \neq \emptyset. \)

**Theorem** (E. Granirer, Illinois J. Math., 1963, M. Klawe, TAMS, 1977). Let \( S \) be a left amenable semigroup. Then \( \dim \langle F(S) \rangle = n \iff S \) contains exactly \( n \) finite left ideal group

In particular, if \( S \) is a group, then \( \dim \langle F(S) \rangle < \infty \iff S \) is a finite group.

In this case, \( \ell^\infty(S) \) has exactly one left invariant mean.
Theorem 1 (Lau, 1976, PAMS). Let $E$ be a dual Banach space with a fixed predual $E_*$ i.e. $(E_*)^* = E$. Let $S$ be a semigroup of linear operators from $E$ into $E$ satisfying:

(i) $\|sx\| \leq \|x\|$ for all $x \in E$, $s \in S$.

(ii) Each $s \in S$ is a continuous linear map from $(E, \text{weak}^*)$ into $(E, \text{weak}^*)$.

Let $\mathcal{G} \subseteq \overline{\co}^W(S)$ be a closed subsemigroup of $\overline{\co}^W(S)$ in the $W^*OT$ (weak* operator topology). Let $X$ be a $\mathcal{G}$-invariant subset of $E$. Let $K$ be a weak* -closed subset of $F(S) = \{x \in E; s(x) = x \text{ for all } s \in S\}$. If $\mathcal{G}(x) \cap K \neq \emptyset$ for each $x \in X$, then there exist $P \in \mathcal{G}$ such that for each $x \in X$, $P(x) \in K$. 
A translation invariant subspace $X \subseteq \ell^\infty(S)$ is called *left introverted* (M.M. Day 1957) if for each $f \in X$, $\phi \in \ell^\infty(S)^*$, the function $\phi\ell(f)(s) = \langle \phi, \ell_s f \rangle$, $s \in S$, is in $X$.

**Example.** Let $S$ be a semitopological semigroup i.e. $S$ is a semigroup with a topology such that for each $a \in S$, the mappings $s \to a \cdot s$, and $s \to s \cdot a$ are continuous. Let $LUC(S) = \{ f \in CB(S); s \to \ell_s f \text{ is continuous when } CB(S) \text{ has the } \| \cdot \|_\infty \text{-topology} \}$. Then $LUC(S)$ is left introverted.

**Corollary** (Granirer-Lau, 1971, Illinois J. Math.). Let $X$ be a translation invariant left introverted subspace of $\ell^\infty(S)$. Then $X$ has a left invariant mean $\iff$

(*) for each $f \in X$, there exist $m \in X^*$, (depending on $f$), $\| m \| = m(1) = 1$, and $\langle m, \ell_a f \rangle = m(f)$ for all $a \in S$.

**Open problem 1:** Does condition (*) imply $X$ has a left invariant mean when $X$ is not left introverted?

**Example.** The space $CB(S)$ is in general not left introverted in general.
3. Locally compact groups

A topological group $(G, T)$ is a group $G$ with a Hausdorff topology $T$ such that

(i) $G \times G \rightarrow G$

$(x, y) \rightarrow x \cdot y$

(ii) $G \rightarrow G$

$x \rightarrow x^{-1}$

are continuous.

$G =$ topological group then

$LUC(G) = \text{all } f \in CB(G)$

such that for any $\varepsilon > 0$, there exists a neighbourhood $U$ of the identity $e$ such that

$|f(x) - f(y)| < \varepsilon$ \hspace{1em} \text{if} \hspace{1em} yx^{-1} \in V, \hspace{1em} x, y \in G$

(i.e. $f$ is uniformly continuous with respect to the right uniformity of $G$).
A topological group $G$ is (left) amenable if $LUC(G)$ has a left invariant mean.


$G = \text{group of unitary operators on a Hilbert space with strong operator topology.}$

Then $G$ is extremely amenable i.e. $LUC(G)$ has a multiplication left invariant mean.

In the case $G$ has the following fixed point property (T. Mitchell, Trans. A.M.S. 1970): Whenever $G$ acts on a compact $T_2$-space $X$, then $X$ contains a common fixed point for $G$.

(See Lau-Ludwig: “Fourier Stieltjes algebra on topological groups”, Advances of Math. 2012 for more examples.)


$G$ is *locally compact* if the topology $\mathcal{T}$ is locally compact i.e. there is a basis for the neighbourhood system of the identity consisting of compact sets.

Ex: $G_d, \mathbb{IR}^n, (E, +), \mathbb{T}, \mathbb{Q}, GL(2, \mathbb{IR}), E = \text{Banach space}, \mathbb{T} = \{\lambda \in \mathbb{C}; |\lambda| = 1\}$

Note that: $(\mathbb{Q}, +)$ is not locally compact.

$(E, +)$ is locally compact if and only if $E$ is finite dimensional.

**Examples:** Compact groups, abelian groups are amenable.

Let $G$ be a *locally compact group*.

- There is a left Haar measure $\lambda$ on $G$ i.e. $\lambda(E) = \lambda(gE)$ for each $g \in G$ and Borel subset $E \subseteq G$.

- “$\lambda$” is unique up to multiplication by positive constant.

**Examples:**

- If $G = (\mathbb{IR}, +)$, $\lambda = \text{Lebesgue measure}$.

- If $G = \text{discrete group}$, $\lambda = \text{counting measure}$

  i.e. $\lambda(E) = |E|$ if $E$ is finite, $\lambda(E) = \infty$ if $E$ is infinite.
Theorem 2 (Graniner and Lau 1971). Let $G$ be a locally compact group. If $G$ is extremely amenable, then $G = \{e\}$.

A map $S : X \to X$ is affine-linear if there exist a (complex) linear map $S^1(x) : X \to X$ as well as an element $x_S \in X$ such that $S(x) = S^1(x) + x_S$ ($x \in X$).

The following theorem can be regarded as a variant of Day’s fixed point theorem of amenable groups concerning affine-linear actions on a dual Banach space $E^*$ rather than affine actions on weak*-compact convex subsets of $E^*$.

Theorem 3 (Chen-Lau-Ng, Studia Math. 2015). $G$ is amenable if and only if any weak*-continuous affine-linear action $\alpha$ of $G$ on any dual Banach space $E^*$ with one norm-bounded orbit (and equivalently, with all orbits being bounded) has a fixed point.

It should be noted that “affine actions” in this result cannot be replaced by “linear actions” since any linear action always has a common fixed point, namely “0”.

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Let $G$ be a locally compact group.

$L^1(G) =$ group algebra of $G$ i.e. $f : G \mapsto \mathbb{C}$ measurable such that

$$\int |f(x)|d\lambda(x) < \infty$$

$$(f * g)(x) = \int f(y)g(y^{-1}x)d\lambda(y)$$

$$\|f\|_1 = \int |f(x)|d\lambda(x)$$

$(L^1(G), \ast)$ is a Banach algebra i.e. $\|f * g\| \leq \|f\| \|g\|$ for all $f, g \in L^1(G)$

$L^\infty(G) =$ essentially bounded measurable functions on $G$.

$$\|f\|_\infty = \text{ess-sup norm.} = \inf \{M : \{x \in G; |f(x)| > M \text{ is a locally null set}\}\}$$

$L^\infty(G)$ is a commutative $C^*$-algebra containing $CB(G)$

$L^1(G)^* = L^\infty(G) : \langle f, h \rangle = \int f(x)h(x)d\lambda(x)$
$G = \text{locally compact abelian group.}$ Then $L^1(G)$ is a \textbf{commutative} Banach algebra.

A complex function $\gamma$ on $G$ is called a \textbf{character} if $\gamma$ is a homomorphism of $G$ into $(\mathbb{T}, \cdot)$.

$$\hat{G} = \text{all continuous characters on } G$$

$$\subseteq L^\infty(G) = L^1(G)^*.$$ 

\textbf{Example}

\begin{align*}
G &= \mathbb{R} \quad \hat{G} = \mathbb{R} \\
G &= \mathbb{T} \quad \hat{G} = \mathbb{Z} \\
G &= \mathbb{Z} \quad \hat{G} = \mathbb{T}.
\end{align*}

Equip $\hat{G}$ with the weak*-topology from $L^1(G)^*$ (or the topology of uniform convergence on compact sets). Then

$$\hat{G} \text{ with product: } (\gamma_1 + \gamma_2)(x) = \gamma_1(x)\gamma_2(x)$$

is a locally compact \textbf{abelian} group called the “dual group” of $G$. 

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Let $G$ be a locally compact group.

$L^2(G) = \{ f : G \rightarrow \mathbb{C} \mid \int |f(x)|^2 d\lambda(x) < \infty \}$

$\langle f, g \rangle = \int f(x) \overline{g(x)} d\lambda(x)$

$L^2(G)$ is a Hilbert space.

Left regular representation:

$\{\rho, L^2(G)\};$

$\rho : G \rightarrow B(L^2(G));$

$\rho(x)h(y) = h(x^{-1}y), \ x \in G, \ h \in L^2(G).$
Let $G$ be a topological group.

A continuous function $\phi : G \to \mathbb{C}$ is **positive definite** if $\forall \lambda_1 \ldots \lambda_n \in \mathbb{C}$, $x_1 \ldots x_n \in G$

\[
\sum_{i,j=1}^{n} \lambda_i \overline{\lambda}_j \phi(x_i x_j^{-1}) \geq 0
\]

$\iff \exists$ a continuous unitary representation $\{\pi, H_\pi\}$ of $G$, $\xi \in H_\pi$, such that

$\phi(x) = \langle \pi(x)\xi, \xi \rangle \ \forall \ x \in G$

$\iff \phi$ is of **positive type** if $G$ is locally compact

i.e. $\langle \phi, f^* \ast f \rangle \geq 0 \ \forall \ f \in L^1(G), f^* = \frac{1}{\Delta(x)} \overline{f(x^{-1})}$

$\Delta = \text{modular function of } G; \ \Delta(x) = \frac{\lambda(Ex)}{\lambda(E)}, \ x \in G,$

$E$ a Borel subset of $G, \ 0 < \lambda(E) < \infty.$
$P(G) = \text{all continuous positive}
\text{definite functions on } G
\subseteq CB(G)$

$(P(G), \cdot)$ is a commutative semigroup with pointwise multiplication.

Let $B(G) = \langle P(G) \rangle$. Then $B(G)$ is a translation invariant subalgebra of $CB(G)$.

$B(G)$ is called the **Fourier Stieltjes algebra** of $G$. 
We now assume that $G$ is locally compact.

$$\mathcal{C}^*(G) = \text{completion of } \left( L^1(G), \| \cdot \| \right)$$

(Group $C^*$-algebra of $G$)

$$\|\|f\|| = \sup \{ \|\pi(f)\| : \pi \text{ continuous unitary representation of } G \} \leq \|f\|_1$$

$$\langle \pi(f)\xi, n \rangle = \int \langle \pi(x)\xi, n \rangle f(x) d\lambda(x) \quad \xi, n \in H_\pi,$$

$f \in L^1(G)$.

If $G$ is amenable, then

$$\|\|f\|| = \|\rho(f)\|, \quad \text{where } \rho(x)h(y) = h(x^{-1}y), \ h \in L^2(G).$$
For \( \phi \in B(G) = C^*(G)^* \)

\[
\|\phi\| = \sup \{ |\langle \phi, f \rangle| : f \in L^1(G), \|f\| \leq 1 \}
\]

\[\geq \|\phi\|_\infty \quad \text{(sup norm)}\]

\((B(G), \| \cdot \|)\) is a commutative Banach algebra

\[
A(G) = \overline{B(G) \cap C_c(G)}^{\| \cdot \|_{B(G)}} \subseteq B(G)
\]

(Fourier algebra of \( G \)) where \( C_c(G) \) are continuous functions \( f : G \to \mathbb{C} \) with compact support.

\( A(G) \) is called the **Fourier algebra** of \( G \).

\[
A(G) = \{ \text{All } \phi : G \to \mathbb{C} \text{ of the form } \phi(x) = \langle \rho(x) h, k \rangle, \ h, k \in L^2(G), \ x \in G \} \subseteq C_0(G).
\]
\[- A(G) \text{ is a closed ideal in } B(G) \]
\[- A(G)^* = VN(G) \]
\[= \langle \rho(x) : x \in G \rangle^{\text{wot}} \subseteq B(L_2(G)) \]

When \( G \) is abelian, then \( A(G) \cong L^1(\hat{G}) \quad VN(G) \cong L^\infty(\hat{G}) \)
\[B(G) \cong M(\hat{G}) \quad C^*(G) \cong C_0(\hat{G}) \]

**Example.** \( G = \mathbb{T} = \{ \lambda \in \mathbb{C} : |\lambda| = 1 \} \), \( \hat{G} = (\mathbb{Z}, +) \).
\[A(\mathbb{T}) \cong \ell^1(\mathbb{Z}), \quad VN(\mathbb{T}) = \ell^\infty(\mathbb{Z}) \]
\[B(\mathbb{T}) \cong M(\mathbb{Z}) = \ell^1(\mathbb{Z}), \quad C^*(\mathbb{T}) \cong c_0(\mathbb{Z}). \]


4. Fixed point set in the group von Neumann algebra $VN(G)$

For $\sigma \in B(G)$, $T \in VN(G)$, define $\sigma \cdot T \in VN(G)$

$$\langle \sigma \cdot T, \psi \rangle = \langle T, \sigma \psi \rangle, \quad \psi \in A(G).$$

Let $I_\sigma = \{\sigma \phi - \phi : \phi \in A(G)\}^{\|\cdot\|}$

$$\subseteq A(G).$$

Then

(i) $I_\sigma$ is a closed ideal in $A(G)$

(ii) $I_\sigma^\perp = \{T \in VN(G) : \sigma \cdot T = T\} =$ fixed point set of $\sigma$


is a weak*-closed subspace of $VN(G)$.)
(iii) If \( \sigma \in P^1(G) = \{ \phi \in P(G); \phi(e) = 1 \} \), then \( H = \{ x \in G; \sigma(x) = 1 \} \) is a closed subgroup of \( G \), and

\[
\text{Fixed point set of } \sigma = I_{\sigma} = VN_H(G) = \langle \rho(h) : h \in H \rangle^{w^*} \subseteq VN(G).
\]

In particular \( I_{\sigma} \) is a \( W^* \)-subalgebra of \( VN(G) \).

**Theorem** (Choquet-Deny 1960). Let \( G \) be a locally compact abelian group. \( \mu \) be a probability measure on \( G \), i.e. \( \mu \in M(G) \), \( \mu \geq 0 \) and \( \| \mu \| = 1 \). Then TFAE:

(i) support \( \mu \) generates a dense subgroup in \( G \)

(ii) any \( \mu \)-harmonic function \( h \in L^\infty(G) \) is constant i.e. \( \mu * h = h \) implies \( h = \lambda \cdot 1 \) for some \( \lambda \in \mathfrak{C} \).
Let $\sigma \in B(G)$,

$$I_\sigma^\perp = \{T \in VN(G) : \sigma \cdot T = T\}.$$ 

**Theorem** (Choquet and Deny). Let $G$ be a locally compact **abelian** group TFAE for $\sigma \in P_1(G)$

(i) For $T \in VN(G)$, $\sigma \cdot T = T \implies T = \lambda I$

(ii) $\sigma(g) \neq 1$ if $g \neq e$.

**Theorem** (Granirer 86). Let $\sigma \in B(G)$

(a) $\dim(I_\sigma^\perp) = n < \infty \iff \{g \in G : \sigma(g) = 1\}$ is a finite set.

   In this case, $\dim(I_\sigma^\perp) = |\{g \in G : \sigma(g) = 1\}|$.

(b) If $G$ is amenable, $I_\sigma^\perp$ is reflexive $\implies I_\sigma^\perp$ is finite dimensional.
Theorem (Granirer 86). Let $S \subseteq B(G)$ be a norm bounded semigroup and

$$F = \{ T \in VN(G); \sigma \cdot T = T \text{ for all } \sigma \in S \}.$$ 

Then there exists a bounded linear projection $P$ from $VN(G)$ onto $F$ such that

$$P(\phi \cdot T) = \phi \cdot P(T) \text{ for all } \phi \in A(G).$$

In particular if $\sigma \in B(G), \|\sigma\| = 1,$ there exists a contractive projection $P : VN(G)$ onto $I_\sigma^\perp$ satisfying

$$P(\phi \cdot T) = \phi \cdot P(T) \text{ for all } \phi \in A(G).$$

Proof. Apply Markov-Kakutani Fixed Point Theorem and Theorem 1.
Let $A$ be a commutative Banach algebra with a BAI.

For $f \in A^*$ and $a \in A$, by $a \cdot f$ we denote the functional on $A$ defined by $\langle a \cdot f, b \rangle = \langle f, ab \rangle$.

A projection $P : A^* \to A^*$ is said to be “invariant” (or $A$-invariant) if, for an $a \in A$ and $f \in A^*$, the equality $P(a \cdot f) = a \cdot P(f)$ holds. Similarly, a closed subspace $X$ of $A^*$ is said to be “invariant” if, for each $a \in A$ and $f \in X$, the functional $a \cdot f$ is in $X$ (i.e. $X$ is an $A$-module for the natural action $(a, f) \mapsto a \cdot f$). If there is an invariant projection from $A^*$ onto a closed invariant subspace $X$ of $A^*$ then $X$ is said to be “invariantly complemented in $A^*$”.

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We say that a projection $P : A^* \hookrightarrow A^*$ is “natural” if, for each $\gamma \in \Delta(A)$, either $P(\gamma) = \gamma$ or $P(\gamma) = 0$.

If $X$ is a closed invariant subspace of $A^*$ and if there is natural projection $P$ from $A^*$ onto $X$ we shall say that $X$ is “naturally complemented” in $A^*$.

**Remark:** Every invariant projection $P : A^* \rightarrow A^*$ is natural.

**Theorem 4** (Lau and Ülger, Trans. A.M.S. 2014). Let $G$ be an amenable locally compact group, and $I$ be a closed ideal in $A(G)$. Then $X = I^\perp$ is invariantly complemented in $VN(G) \iff X$ is naturally complemented. In particular, if $\sigma \in B(G)$, $\|\sigma\| = 1$, there is an invariant projection onto $I^\perp_\sigma = \{T \in VN(G) : \sigma \cdot T = T\}$ (the fixed point set of $\sigma$) $\iff I^\perp_\sigma$ is naturally complemented.
5. Metric semigroup

A metric semigroup is a semitopological semigroup whose topology is determined by a metric $d$. We consider the following fixed point property for a metric semigroup $S$.

$$(F_U) : \text{If } S = \{T_s : s \in S\} \text{ is a separately continuous representation of } S \text{ on a compact subset } K \text{ of a locally convex space } (E, Q) \text{ and if the mapping } s \mapsto T_s(y) \text{ from } S \text{ into } K \text{ is uniformly continuous for each } y \in K, \text{ then } K \text{ has a common fixed point for } S.$$

Note that the mapping $s \mapsto T_s(y)$ is uniformly continuous if for each $\tau \in Q$ and each $\varepsilon > 0$ there is $\delta > 0$ such that

$$\tau(T_s(y) - T_t(y)) \leq \varepsilon$$

whenever $d(s, t) \leq \delta$. 
For example, suppose that $S$ is a subset of a locally convex space $L$ that acts on $E$ such that $(a, y) \mapsto ay : L \times E \to E$ is separately continuous; if $a \mapsto ay$ is linear in $a \in L$ for each $y \in E$, then the induced action of $S$ on $E$, $(s, y) \mapsto sy(s \in S)$, is uniformly continuous in $s$ for each $y \in E$.

A Banach algebra $A$ is a $F$-algebra (Lau [1983]) if it is the (unique) predual of a $W^*$-algebra $\mathcal{M}$ and the identity $e$ of $\mathcal{M}$ is a multiplicative linear functional on $A$.

Since $A^{**} = \mathcal{M}^*$, we denote by $P_1(A^{**})$ the set of all normalized positive linear functionals on $\mathcal{M}$, that is,

$$P_1(A^{**}) = \{m \in A^{**} : m \geq 0, m(e) = 1\}.$$ 

In this case $P_1(A^{**})$ is a semigroup with the first (or second) Arens multiplication.
Examples of $F$-algebras include the predual algebras of a Hopf von Neumann algebra (in particular, quantum group algebras), the group algebra $L^1(G)$ of a locally compact group $G$, the Fourier algebra $A(G)$ and the Fourier-Stieltjes algebra $B(G)$ of a topological group $G$. They also include the measure algebra $M(S)$ of a locally compact semigroup $S$.

A mean $m \in P_1(A^{**})$ on $A^* = \mathcal{M}$ is called a topological left invariant mean, abbreviated as TLIM, if $a \cdot m = m$ for all $a \in P_1(A)$: in other words, $m \in P_1(A^{**})$ is a TLIM if $m(x \cdot a) = m(x)$ for all $a \in P_1(A)$ and $x \in A^*$, i.e. $A$ is left amenable.

**Lemma A.** The $F$-algebra $A$ is left amenable if and only if the metric semigroup $P_1(A)$ has the fixed point property ($F_U$).

**Theorem 5** (Lau and Zhang, Trans. A.M.S. 2016). Let $A$ be an $F$-algebra. Then $A$ is left amenable if and only if $P_1(A)$ has the fixed point property ($F_E$).
(\(F_E\)) : Every jointly continuous representation of \(S\) on a non-empty compact Hausdorff space \(C\) has a common fixed point in \(C\) i.e. the semigroup \(P_1(A)\) is extremely left amenable.

**Theorem 6** (A.T.-M. Lau, C.K. Ng and N.C. Wong, Oxford Quarterly Journal of Math. 2018). Two locally compact groups \(G\) and \(H\) are isomorphic as topological groups if and only if any one of the following holds

1. \(L^1(G)^1_+ \cong L^1(H)^1_+\) as metric semigroups;

2. \(M(G)^1_+ \cong M(H)^1_+\) as metric semigroups.

**Open problem 2.** Let \(G\) and \(H\) be locally compact groups, and \(H^{op}\) be the opposite group of \(H\). If there is a metric preserving semigroup isomorphism from \(A(G)^1_+\) (respectively, \(B(G)^1_+\)) onto \(A(H)^1_+\) (respectively, \(B(H)^1_+\)), does either \(G = H\) or \(G = H^{op}\)?
Theorem 7 (A.T.-M. Lau, C.K. Ng and N.C. Wong, 2019).

(a) Let $A_1$ and $A_2$ be two $F$-algebras with $A_1^*$ being a semi-finite $W^*$-algebra. Any metric semi-group isomorphism $\Phi : P_1(A_1^{**}) \to P_1(A_2^{**})$ extends to an isometric isomorphism from $A_1$ onto $A_2$.

(b) Let $G_1$ and $G_2$ be two locally compact groups such that $G_1$ is unimodular. If there is a metric semi-group isomorphism

$$\Psi : P_1(VN(G_1)^*) \to P_1(VN(G_2)^*) \quad \text{or} \quad \Psi : P_1(B(G_1)^{**}) \to P_1(B(G_2)^{**})$$

then there exists either a homeomorphis group isomorphism or a homeomorphic group anti-isomorphism $\Theta : G_2 \to G_1$ such that $\Psi(f) = f \circ \Theta$. 

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6. Fixed point set of power bounded elements in $B(G)$

A (not necessarily linear) mapping $T$ on a Banach space $X$ is said to be non-expansive if $\|T(x) - T(y)\| \leq \|x - y\|$ holds for all $x, y \in X$.

**Theorem** (Ishikawa 1976). Let $X$ be a Banach space, let $T : X \to X$ be a nonexpansive mapping and let $0 < \lambda < 1$. Consider $S_\lambda = \lambda I + (1 - \lambda)T$. Then, for each $x \in X$, we have

$$\lim_{n \to \infty} \|S_\lambda^{-1}(x) - S_\lambda^n(x)\| = 0.$$
Let $T : X \to X$ be a linear operator from a Banach $X$ into $X$. Then $T$ is called **power bounded** if there exists $M > 0$ such that $\|T^n\| \leq M$ for all $n = 1, 2, \ldots$.

**Lemma B.** Let $X$ be a Banach space and let $T : X \to X$ a power bounded linear map. Let $0 < \lambda < 1$ and $S_\lambda = \lambda I + (1 - \lambda)T$. Then, for each $x \in X$, we have

$$\lim_{n \to \infty} \|S_\lambda^{n+1}(x) - S_\lambda^n(x)\| = 0.$$

**Proof.** Let $c = \sup\{\|T^n\| : n \in \mathbb{N}_0\}$ and define a new norm $|\cdot|$ on $X$ by $|x| = \sup \{\|T^n(x)\| : n \in \mathbb{N}_0\}$, $x \in X$. Then $\|x\| \leq |x| \leq c\|x\|$, so the norms $\|\cdot\|$ and $|\cdot|$ on $X$ are equivalent. With respect to the norm $|\cdot|$, the operator norm of $T$ is at most $1$. Thus $T : (X, |\cdot|) \to (X, |\cdot|)$ is nonexpansive. By Ishikawa Theorem, $|S_\lambda^{n+1}(x) - S_\lambda^n(x)| \to 0$ and hence also $\|S_\lambda^{n+1}(x) - S_\lambda^n(x)\| \to 0$ for every $x \in X$. $\square$
Let $E$ be a dual Banach space with a (fixed) predual $E_*$, i.e. $E_*$ is a Banach space such that $(E_*)^* = E$.

Let $\mathcal{B}(E)$ be the space of bounded linear operators from $E$ into $E$. By the weak$^*$-operators topology on $\mathcal{B}(E)$, denoted by $W^*OT$, we shall mean the locally convex topology determined by the family of seminorms $\{p_{x,\phi}; x \in E \text{ and } \phi \in E_*\}$ where $p_{x,\phi}(T) = |\phi(Tx)| \ T \in \mathcal{B}(E)$. Then the unit ball of $\mathcal{B}(E)$ is compact in the $W^*OT$.

**Lemma C.** Let $A$ be a Banach algebra. Let $(S_\alpha)_{\alpha}$ be a bounded net in $B(A^*)$ and let $R, S \in B(A^*)$. Then we have the following:

(i) $S_\alpha \to S$ in the $W^*OT$ if the and only if $\langle S_\alpha(f), a \rangle \to \langle S(f), a \rangle$ for each $f \in A^*$ and $a \in A$;

(ii) if $S_\alpha \to S$ in the $W^*OT$, then $S_\alpha \circ R \to S \circ R$ in the $W^*OT$;

(iii) if $S_\alpha \to S$ in the $W^*OT$, then $T^* \circ S_\alpha \to T^* \circ S$ in the $W^*OT$ for any bounded linear operator $T : A \to A$. 

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Let $A$ be a commutative Banach algebra. A linear operator $T : A \rightarrow A$ is called a **multiplier** of $A$ if $T(ab) = aT(b)$ holds for all $a, b \in A$. When $A$ is faithful i.e. for any $a \in A$, the condition $aA = \{0\}$ implies $a = 0$, then every multiplier of $A$ is bounded and the set $M(A)$ of all multipliers of $A$ is a unital commutative closed subalgebra of $B(A)$ of bounded linear operators on $A$.

- If $A = L^1(G)$, then $M(A) \cong M(G)$ (measure algebra)
- If $A = A(G)$ and $G$ is amenable, then $M(A) \cong B(G)$ (Fourier Stieltjes algebra)

**Theorem 7** (Kaniuth, Lau, Ulger, JLMS 2010). *Given any power bounded multiplier $T$ of a commutative Banach algebra $A$, there exists an $A$-invariant projection $P : A^* \rightarrow A^*$ such that*

$$P \circ T^* = T^* \circ P = P \quad \text{and} \quad P(A^*) = \ker (I - T^*) = \{\phi \in A^*; T^*\phi = \phi\}.$$
**Proof.** Consider $S = \frac{1}{2} (I + T)$. The sequence $((S^*)^n)_{n}$ is a bounded sequence in $B(A^*)$. Then there exists a subnet $((S^*)^{n_\alpha})_{n_\alpha}$ that converges in the $W^*OT$-topology to some $P \in B(A^*)$. Since, for all $a \in A$, $f \in A^*$ and $n \in \mathbb{IN}$, we have

$$|\langle (S^*)^{n+1}(f) - (S^*)^n(f), a \rangle| = |\langle f, S^{n+1}(a) - S^n(a) \rangle|$$

$$\leq \|f\| \cdot \|S^{n+1}(a) - S^n(a)\|,$$

we conclude from Lemmas B and C that $S^* \circ P = P \circ S^* = P$. This implies that $(S^*)^n \circ P = P$ for all $n \in \mathbb{IN}$. Passing to the $W^*OT$ limit in $B(A^*)$, we get that $P^2 = P$. Since $T = 2S - I$, we also get $T^* \circ P = P \circ T^* = P$, and in particular $(I - T^*) \circ P = 0$. This shows that $P^*(A^*) \subseteq \ker(I - T^*)$. To see the reverse inclusion, let $f \in \ker(I - T^*)$ be given. Then $T^*(f) = f$ and so $S^*(f) = f$. Then $(S^*)^n(f) = f$ for all $n$ and by the usual $w^*$-limit argument this gives $P^*(f) = f$. Thus $\ker(I - T^*) \subseteq P^*(A^*)$. \(\square\)
For a discrete group $D$, let $R(D)$ denote the Boolean ring of subsets of $D$ generated by all left cosets of subgroups of $D$.

Let $R_c(G) = \{E \in R(G_d) : E \text{ is closed in } G\}$

$G_d = \text{ denote } G \text{ with the discrete topology.}$

**Theorem** (J. Gilbert, B. Schreiber, B. Forrest). $E \in R_c(G) \iff E = \bigcup_{i=1}^{n} (a_iH_i \setminus \bigcup_{j=1}^{m_i} b_{i,j}K_{ij})$, where $a_i, b_{i,j} \in G$, $H_i$ is a closed subgroup of $G$ and $K_{ij}$ is an open subgroup of $H_i$. 


If \( u \in B(G) \), let

\[
E_u = \{ x \in G; |u(x)| = 1 \}
\]

and

\[
F_u = \{ x \in G; u(x) = 1 \}.
\]

**Remark:** If \( u \in B(G) \) is power bounded, \( w = \frac{1}{2} (1 + u) \), then \( E_w = F_w = F_u \).

**Theorem 8** (Kaniuth-Lau-Ülger 2010, JLMS). Let \( G \) be any locally compact group and \( u \in B(G) \) be power bounded (i.e. \( \sup \{ \| x^n \| ; n = 1, 2, \ldots \} < \infty \)). Then

(a) The sets \( E_u \) and \( F_u \) are in \( R_c(G) \).

(b) The fixed point set of \( u \) in \( VN(G) = \{ T \in VN(G); u \cdot T = T \} \) is the range of a projection \( P : VN(G) \to VN(G) \) such that \( u \cdot P(T) = P(u \cdot T) \) for all \( T \in VN(G) \). If \( G \) is amenable, then \( \{ T \in VN(G); u \cdot T = T \} = \overline{\langle \rho(x); x \in F_u \rangle}^{W^*} = \overline{\langle \rho(x); x \in E_w \rangle}^{W^*} \) where \( w = \frac{1}{2} (1 + u) \).
7. **Weak*-fixed point property for $B(G)$**

- A dual Banach space $E$ is said to have the **weak*-fixed point property** if for each weak*-compact compact convex subset of $E$ has the fixed point property for non-expansive mappings.

- We say that a dual Banach space $E$ has the **weak* f.p.p. for left reversible semigroup** if whenever $S$ is a left reversible semigroup and $K$ is a weak* compact convex subset of $E$ and $S = \{T_s : s \in S\}$ is a representation of $S$ as non-expansive mappings from $K$ into $K$, then $K$ has a common fixed point for $S$.

**Theorem** (T.C. Lim, Pacific J. Math. 1980). $\ell_1 = c_0^*$ has the weak*-f.p.p. property.

Theorem (N. Randrianantoanina, JFA 2010). Let $H$ be a Hilbert space. Then $B(H)_*$ has the weak* f.p.p. for left reversible semigroups.

Let $C$ be a non-empty subset of a Banach space $X$ and $\{D_\alpha|\alpha \in \Lambda\}$ be a decreasing net of bounded non-empty subsets of $X$. For each $x \in C$, and $\alpha \in \Lambda$, let

$$r_\alpha(x) = \sup\{\|x - y\| | y \in D_\alpha\}$$

$$r(x) = \lim_{\alpha} r_\alpha(x) = \inf_{\alpha} r_\alpha(x)$$

$$r = \inf\{r(x)|x \in C\}.$$ 

The set (possibly empty)

$$\mathcal{AC}(\{D_\alpha|\alpha \in \Lambda\}) = \{x \in C|r(x) = r\}$$

is called the asymptotic centre of $\{D_\alpha|\alpha \in \Lambda\}$ with respect to $C$ and $r$ is called the asymptotic radius of $\{D_\alpha|\alpha \in \Lambda\}$ with respect to $C$. 

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**Theorem 9** (Fendler-Lau-Leinert JFA 2013). Let $G$ be a compact group. Let $C$ be a nonempty weak$^*$ closed convex subset of $B(G)$ and $\{D_\alpha : \alpha \in \Lambda\}$ be a decreasing net of nonempty bounded subsets of $C$. Let $r(x)$ be as defined above. Then for each $s \geq 0$, $\{x \in C : r(x) \leq s\}$ is weak$^*$ compact and convex, and the asymptotic center of $\{D_\alpha : \alpha \in \Lambda\}$ with respect to $C$ is a nonempty norm compact convex subset of $C$.

**Corollary.** Let $G$ be a compact group, then $B(G) = A(G)$ has the weak$^*$-f.p.p. for left reversible semigroups.

- $G$-separable (Lau-Mah 2010 JFA)
- Corollary above can be derived from a result in [N. Randrainantoanina JFA 2010] using non-commutative integration theory.
**Theorem 10** (Fendler-Lau-Leinert JFA 2013). Let $G$ be a locally compact group. Then $B(G)$ has the $w^*$-fixed point property if and only if $G$ is compact. In this case, $B(G)$ has the weak*-fixed point property for left reversible semigroups.

**Open problem 3.** Let $G$ be a locally compact group. Let $B_\rho(G)$ denote the reduced Fourier-Stieltjes algebra of $B(G)$, i.e. $B_\rho(G)$ is the weak* closure of $C_{00}(G) \cap B(G)$. Then $B_\rho(G) = C_\rho^*(G)^*$ where $C_\rho^*(G) = \text{norm closure of } \{ \rho(f) : f \in L^1(G) \}$ in $VN(G)$. Does the weak* fixed point property on $B_\rho(G)$ imply $G$ is compact? This is true when $G$ is amenable by Theorem 10, since $B(G) = B_\rho(G)$ in this case.

**Open problem 4.** Let $G$ be a locally compact group. Does the asymptotic centre property on $B_\rho(G)$ imply that $G$ is compact?
References


