Holomorphic functional calculus for finite families of commuting semigroups

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The author observed in [8] that if a Banach algebra $A$ does not possess any nonzero idempotent then \[ \inf_{x \in A, \|x\| \geq 1/2} \|x^2 - x\| \geq 1/4. \] If $x$ is quasinilpotent, and if \( \|x\| \geq 1/2 \), then \( \|x\| > 1/4 \). Concerning (nonzero) strongly continuous semigroups $T = (T(t))_{t > 0}$ of bounded operators on a Banach space $X$, these elementary considerations lead to the following results, obtained in 1987 by Mokhtari

- If \( \limsup_{t \to 0^+} \|T(t) - T(2t)\| < 1/4 \), then the generator of the semigroup is bounded, and so \( \limsup \|T(t) - T(2t)\| = 0 \).
- If the semigroup is quasinilpotent, then \( \|T(t) - T(2t)\| > 1/4 \) when $t$ is sufficiently small.

If the semigroup is norm continuous, and if there exists a sequence $(t_n)_{n \geq 1}$ of positive real numbers such that \( \lim_{n \to +\infty} t_n = 0 \) and \( \|T(t_n) - T(2t_n)\| < 1/4 \), then the closed subalgebra $\mathcal{A}_T$ of $\mathcal{B}(X)$ generated by the semigroup possesses an exhaustive sequence of idempotents, i.e. there exists a sequence $(P_n)_{n \geq 1}$ of idempotents of $\mathcal{A}_T$ such that for every compact set $K \subset \hat{\mathcal{A}}_T$ there exists $n_K > 0$ satisfying $\chi(P_n) = 1$ for every $\chi \in K$, $n \geq n_K$. 
More sophisticated arguments allowed A. Mokhtari and the author to obtain in 2002 the following more general results [12], valid for every integer $p \geq 1$,

1. If $\limsup_{t \to 0^+} \| T(t) - T((p + 1)t) \| < \frac{p}{(p+1)^{1+1/p}}$, then the generator of the semigroup is bounded, and so $\limsup \| T(t) - T((p + 1)t) \| = 0$.

2. If the semigroup is quasinilpotent, then $\| T(t) - T((p + 1)t) \| > \frac{p}{(p+1)^{1+1/p}}$ when $t$ is sufficiently small.

3. If the semigroup is norm continuous, and if there exists a sequence $(t_n)_{n \geq 1}$ of positive real numbers such that $\lim_{n \to +\infty} t_n = 0$ and $\| T(t_n) - T((p + 1)t_n) \| < \frac{p}{(p+1)^{1+1/p}}$ for $n \geq 1$, then the closed subalgebra $\mathcal{A}_T$ of $B(X)$ generated by the semigroup possesses an exhaustive sequence of idempotents.
These results led the author to consider in [9] the behavior of the distance $\| T(s) - T(t) \|$ for $s > t$ near 0. The following results were obtained in [9]

- If there exist for some $\delta > 0$ two continuous functions $r \rightarrow t(r)$ and $r \rightarrow s(r) > t(r)$ on $[0, \delta]$ such that
  \[
  \| T(t(r)) - T(s(r)) \| < (s(r) - t(r)) \frac{s(r)}{t(r)} \frac{s(r) - t(r)}{s(r) - t(r)},
  \]
  then the generator of the semigroup is bounded, and so $\| T(t) - T(s) \| \rightarrow 0$ as $0 < t < s, s \rightarrow 0^+$.  

- If the semigroup is quasinilpotent, there exists $\delta > 0$ such that
  \[
  \| T(t) - T(s) \| > (s - t) \frac{s}{t} \frac{s - t}{s - t} \]
  for $0 < t < s \leq \delta$. 

- If the semigroup is norm continuous, and if there exists two sequences of positive real numbers such that $0 < t_n < s_n$, $\lim_{n \rightarrow +\infty} s_n = 0$, such that
  \[
  \| T(t_n) - T(s_n) \| < (s_n - t_n) \frac{s_n}{t_n} \frac{s_n - t_n}{s_n - t_n},
  \]
  then the closed subalgebra $\mathcal{A}_T$ of $\mathcal{B}(X)$ generated by the semigroup possesses an exhaustive sequence of idempotents.
The quantities appearing in these statements are not mysterious: consider the Hilbert space $L^2([0,1])$, and for $t > 0$ define $T_0(t) : L^2([0,1]) \rightarrow L^2([0,1])$ by the formula $T_0(t)(f)(x) = x^t f(x)$ ($0 < x \leq 1$). Then $\|T_0(t) - T_0(s)\| = (s - t) \frac{s}{t} \frac{s-t}{t}$. This remark also shows that the assertions (1) and (3) in these statements are sharp, and examples show that the assertions (2) are also sharp.

One can consider $T(t)$ as defined by the formula $\int_0^{+\infty} T(x) d\delta_t(x)$, where $\delta_t$ denotes the Dirac measure at $t$. Heuristically, $T(t) = e^{t\Delta}$, where $\Delta$ denotes the generator of the semigroup, and since the Laplace transform of $\delta_t$ is defined by the formula $\mathcal{L}(\delta_t)(z) = \int_0^{+\infty} e^{-zx} d\delta_t(x) = e^{-zt}$, it is natural to write $\mathcal{L}(\delta_t)(-\Delta) = T(t)$. More generally, if an entire function $F$ has the form $F = \mathcal{L}(\mu)$, where $\mu$ is a measure supported by $[a, b]$, with $0 < a < b < +\infty$, we can set

$$F(-\Delta) = \int_0^{+\infty} T(x) d\mu(x),$$

and consider the behavior of the semigroup near 0 in this context.
I. Chalendar, J.R. Partington and the author used this point of view in [4]. Denote by $\mathcal{M}_c(0, +\infty)$ the set of all measures $\mu$ supported by some interval $[a, b]$, where $0 < a < b < +\infty$. For the sake of simplicity we restrict attention to statements analogous to assertion 2. We have the following result

**Quasinilpotent semigroups**

Theorem: Let $\mu \in \mathcal{M}_c(0, +\infty)$ be a nontrivial real measure such that $\int_0^{+\infty} d\mu(t) = 0$ and let $T = (T(t))_{t>0}$ be a quasinilpotent semigroup of bounded operators. Then there exists $\delta > 0$ such that $\|F(-s\Delta)\| > \max_{x\geq 0} |F(x)|$ for $0 < s \leq \delta$.

When $\mu = \delta_1 - \delta_2$ this gives assertion 3 of Mokhtari’s result, and when $\mu = \delta_1 - \delta_{p+1}$ this gives assertion 3 of the Esterle-Mokhtari result (but to obtain extensions of the results of [9] one would need several variables extensions of this functional calculus).

This theory applies, for example, to quantities of the form $\|T(t) - 2T(2t) + T(3t)\|$, or Bochner integrals $\|\int_1^2 T(tx)dx - \int_2^3 T(tx)dx\|$, which are not accessible by the methods of [18] or [12]. Partial results concerning semigroups holomorphic in a sector will appear in [4]. This was the motivation to consider the following question
question 1
Find a good approach to the definition of $F(-\Delta_T)$, where $F$ belongs to a suitable class of holomorphic functions on a suitable class of open subsets of the complex plane and where $\Delta_T$ is the generator of a semigroup $(T(t))_{t>0}$ of bounded operators satisfying some suitable continuity properties. Extend this approach to define $F(-\Delta_{T_1}, \ldots, -\Delta_{T_k})$ where $F$ belongs to a suitable class of holomorphic functions of several complex variables and where $T_1, \ldots, T_k$ is a family of commuting semigroups satisfying some suitable conditions.

question 0
Find a good general notion of continuity for semigroups $(T(t))_{t>0}$ of bounded operators leading to a notion of generator suitable for functional calculus.

An answer to question 0 was given in the Proceedings of the Oulu conference on Banach algebras and applications [10], where one-parameter semigroups of bounded operators on a Banach space $X$ which are weakly continuous with respect to an "Arveson pair" $(X, X_*)$ are studied. An answer to question 1 was recently given by the author in the long paper [11]. The remainder of the talk will be devoted to an outline of the theory developed in [11].
We will use the notion of quasimultipliers introduced by the author in the second of the Long Beach Conference papers [8]: Let $\mathcal{A}$ be a weakly cancellative commutative Banach algebra having dense principal ideals, which means that $\mathcal{A}^\perp := \{a \in \mathcal{A} \mid ax = 0 \forall x \in \mathcal{A}\}$ reduces to $\{0\}$ and that the set $\Omega(\mathcal{A})$ of elements $a \in \mathcal{A}$ such that $a\mathcal{A}$ is dense in $\mathcal{A}$ is nonempty.

**Quasimultipliers**

A quasimultiplier on $\mathcal{A}$ is a pair $(S_{u/v}, D_{u/v})$, where $u \in \mathcal{A}$, $v \in \Omega(\mathcal{A})$ where $D_{u/v}$ is the space of all $x \in \mathcal{A}$ such that $ux \in v\mathcal{A}$, and where the closed operator $S_{u/v} : D_{u/v} \to \mathcal{A}$ is defined by the formula $vS_{u/v}x = ux$.

The algebra of quasimultipliers on $\mathcal{A}$ is denoted $QM(\mathcal{A})$, and a family $U \subset QM(\mathcal{A})$ is said to be pseudobounded if there exists $a \in \Omega(\mathcal{A})$ such that $\sup_{S \in U} \|Sa\| < +\infty$.

**Regular quasimultipliers**

A quasimultiplier $S \in QM(\mathcal{A})$ is said to be regular if there exists $\lambda > 0$ such that the family $(\lambda^n S^n)_{n \geq 1}$ is pseudobounded.

The algebra $QM_r(\mathcal{A})$ of all regular quasimultipliers on $\mathcal{A}$ is a pseudo-Banach algebra in the sense of Allan, Dales and McClure [1].
Recall that a multiplier on $\mathcal{A}$ is a bounded operator $T = \mathcal{A} \to \mathcal{A}$ such that $Tuv = (Tu)v$ for $u, v \in \mathcal{A}$. We denote by $\mathcal{M}(\mathcal{A})$ the algebra of all multipliers on $\mathcal{A}$, identified to the set of all $S = S_{u/v} \in \mathcal{QM}(\mathcal{A})$ such that $\mathcal{D}_{u/v} = \mathcal{A}$, so that $\mathcal{M}(\mathcal{A}) \subset \mathcal{QM}_r(\mathcal{A})$. For $\alpha < \beta \leq \alpha + \pi$ we denote by $S_{\alpha,\beta}$ the open sector \( \{ z \in \mathbb{C} \setminus \{0\} | \alpha < \arg(z) < \beta \} \) and by $\overline{S}_{\alpha,\beta}$ its closure, with the obvious convention $\overline{S}_{\alpha,\alpha} = \{ te^{i\alpha} \}_{t \geq 0}$. We denote by $\mathcal{M}(\mathcal{A}) \subset \mathcal{QM}_r(\mathcal{A})$ the algebra of all multipliers on $\mathcal{A}$. We will consider

two kinds of semigroups of multipliers

(1) Strongly continuous semigroups $T = (T(te^{i\alpha}))_{t > 0}$ of multipliers such that $\bigcup_{t > 0} T(te^{i\alpha}) \mathcal{A}$ is dense in $\mathcal{A}$.

(2) When $\alpha < \beta$, holomorphic semigroups $T(\zeta)_{\zeta \in S_{\alpha,\beta}}$ of multipliers such that $\bigcup_{\zeta \in S_{\alpha,\beta}} T(\zeta) \mathcal{A}$ is dense in $\mathcal{A}$ (which is equivalent to the fact that $T(\zeta) \mathcal{A}$ is dense in $\mathcal{A}$ for some, or for every $\zeta \in S_{\alpha,\beta}$).
**QM-homomorphisms**

Definition: A $QM$-homomorphism $\phi : A \to B$ is a one-to-one homomorphism from $A$ into a Banach algebra $B$ which satisfies the two following conditions

(i) $\phi(A)$ is dense in $B$,
(ii) $\phi(u)B \subset \phi(A)$ for some $u \in \Omega(A)$.

In this situation we will say that $\phi$ is a $QM$-homomorphism with respect to $u$.

**Isomorphisms of quasimultiplier algebras**

Proposition: Assume that $\phi : A \to B$ is a $QM$-homomorphism with respect to $u_0 \in \Omega(A)$. Then the map $\tilde{\phi} : S_{u/v} \to S_{\phi(u)/\phi(v)}$ is a pseudobounded isomorphism from $QM(A)$ onto $QM(B)$, and

$\tilde{\phi}^{-1}(S_{u/v}) = S_{\phi^{-1}(u\phi(u_0))/\phi^{-1}(v\phi(u_0))}$ for $u \in B$, $v \in \Omega(B)$.

A routine verification shows that if $\phi : A \to B$ and $\psi : B \to C$ are $QM$-homomorphisms, then $\psi \circ \phi : A \to C$ is also a $QM$-homomorphism. Every character $\chi$ on $A$ has an obvious extension $\tilde{\chi}$ to $QM(A)$. This allows to define the spectrum $\sigma_A(S) := \{\tilde{\chi}(S)\}_{\chi \in \hat{A}}$ where $\hat{A}$ denotes the set of characters on $A$ equipped with the usual Gelfand topology, with the convention $\sigma_A(S) = \emptyset$ if $A$ is radical.
It is easy to check that if $S - \lambda_0 I$ has an inverse $(S - \lambda_0 I)^{-1}$ in $QM(A)$ which belongs to $A$ for some $\lambda_0 \in \mathbb{C}$, then $\sigma_A(S)$ is closed and $(S - \lambda I)^{-1}$ has an inverse $(S - \lambda I)^{-1} \in A$ for every $\lambda \in \text{Res}_A(S) := \mathbb{C} \setminus \sigma_A(S)$.

**Normalization of $A$ with respect to a strongly continuous semigroup**

**Definition:** Let $T = ((T(t)))_{t > 0}$ be a strongly continuous semigroup of multipliers on $A$ such that $T(t)A$ is dense in $A$ for $t > 0$. A normalization $B$ of $A$ with respect to $T$ is a subalgebra $B$ of $QM(A)$ which is a Banach algebra with respect to a norm $\|\cdot\|_B$ and satisfies the following conditions

(i) The inclusion map $j : A \to B$ is a $QM$-homomorphism, and $\|j(u)\|_B \leq \|u\|_A$ for $u \in A$.

(ii) $\tilde{j}(R) \subset M(B)$, and $\|\tilde{j}(R)\|_{M(B)} \leq \|R\|_{M(A)}$ for $R \in M(A)$, where $\tilde{j} : QM(A) \to QM(B)$ is the pseudobounded isomorphism associated to $j$ introduced in proposition 2.2 (ii).

(iii) $\limsup_{t \to 0^+} \|\tilde{j}(T(t))\|_{M(B)} < +\infty$.

It is always possible to construct a normalization of $A$ with respect to a strongly continuous semigroup $T$. 
For semigroups holomorphic in a sector, a more precise construction is needed.

**Normalization of \( \mathcal{A} \) with respect to a holomorphic semigroup**

Definition: Let \( T := (T(\zeta))_{\zeta \in S_{a,b}} \) be an analytic semigroup of multipliers on \( \mathcal{A} \) such that \( T(\zeta) \mathcal{A} \) is dense in \( \mathcal{A} \) for \( \zeta \in S_{a,b} \). A normalization of the algebra \( \mathcal{A} \) with respect to the semigroup \( T \) is a subalgebra \( \mathcal{B} \) of \( Q\mathcal{M}(\mathcal{A}) \) which is a Banach algebra with respect to a norm \( \| \cdot \|_B \) and satisfies the following conditions

(i) There exists \( u \in \Omega(\mathcal{A}) \) such that the inclusion map \( j : \mathcal{A} \to \mathcal{B} \) is a \( Q\mathcal{M} \)-homomorphism with respect to \( T(\zeta)u \) for every \( \zeta \in S_{a,b} \), and \( \| j(u) \|_B \leq \| u \|_A \) for \( u \in \mathcal{A} \).

(ii) \( \tilde{j}(R) \subset \mathcal{M}(\mathcal{B}) \), and \( \| \tilde{j}(R) \|_{\mathcal{M}(\mathcal{B})} \leq \| R \|_{\mathcal{M}(\mathcal{A})} \) for \( R \in \mathcal{M}(\mathcal{A}) \), where \( \tilde{j} : Q\mathcal{M}(\mathcal{A}) \to Q\mathcal{M}(\mathcal{B}) \) is the pseudobounded isomorphism associated to \( j \).

(iii) \( \limsup_{t \to 0} \| \tilde{j}(T(t)) \|_{\mathcal{M}(\mathcal{B})} < +\infty \) for \( a < \alpha < \beta < b \).

Again, it is always possible to construct a normalization of \( \mathcal{A} \) with respect to a holomorphic semigroup.
Let $T = (T(te^{i\alpha})_{t>0}$ be a strongly continuous of multipliers on $\mathcal{A}$. Set $\omega_T := \|T(te^{i\alpha})\|$, so that $\omega_T$ is lower semicontinuous. Denote by $\mathcal{M}_{\omega_T}$ the convolution algebra of all measures $\mu$ on $(0, +\infty)$ such that $\int_0^{+\infty} \omega_T(t)d|\mu|(t) < +\infty$, and define $\Phi_T : \mathcal{M}_{\omega_T} \to \mathcal{M}(\mathcal{A})$ by using the formula

$$< \Phi_T(\mu)u > = \int_0^{+\infty} T(te^{i\alpha})ud\mu(t) \quad (\mu \in \mathcal{M}_{\omega_T}, u \in \mathcal{A}).$$

Let $L^1_{\omega_T}$ be the space of all (classes of) measurable functions on $(0, +\infty)$ such that $\int_0^{+\infty} |f(t)|\omega_T(t)dt < +\infty$, identified to the closed ideal of $\mathcal{M}_{\omega_T}$ of measures $\mu \in \mathcal{M}_{\omega_T}$ which are absolutely continuous with respect to Lebesgue measure.

The "Arveson ideal" $\mathcal{I}_T$ is the closure in $\mathcal{M}(\mathcal{A})$ of $\Phi_T(L^1_{\omega_T})$. It is a closed ideal of the "Arveson algebra" $\mathcal{A}_T$ defined to be the closure of $\Phi_T(\mathcal{M}_{\omega_T})$ in $\mathcal{M}(\mathcal{A})$. Of course, if the semigroup is analytic on some sector $S_{a,b}$, then $\mathcal{I}_T = \mathcal{A}_T$ is the closed subalgebra of $\mathcal{M}(\mathcal{A})$ generated by semigroup, which is also the closed subspace of $\mathcal{M}(\mathcal{A})$ spanned by $\{T(te^{i\alpha})\}_{t>0}$ for $a < \alpha < b$.

A result of [8] about existence of dense principal ideals in the convolution algebras $L^1_\omega$ extends to the case where the submultiplicative weight $\omega$ is lower semicontinuous, and $\phi_T(f)u \in \Omega(\mathcal{A})$ for $f \in \Omega(L^1_{\omega_T})$, $u \in \Omega(\mathcal{A})$. 

**The Arveson ideal**
We now interpret the generator of a Arveson weakly continuous semigroup as a quasimultiplier on the Arveson ideal $\mathcal{I}_T$.

**The generator as a quasimultiplier on the Arveson ideal**

We can as above consider the generator of $T$ as a quasimultiplier on $\mathcal{I}_T$

Definition: The infinitesimal generator $\Delta_{T,\mathcal{I}_T}$ of $T$ on $\mathcal{I}_T$ is the quasimultiplier on $\mathcal{I}_T$ defined by the formula

$$\Delta_{T,\mathcal{I}_T} = e^{-i\alpha S_{\phi_T(f'_0)/\phi_T(f_0)}},$$

where $f_0 \in C^1([0, +\infty)) \cap \Omega \left( L^1_{\omega_T} \right)$ satisfies $f_0 = 0$, $f'_0 \in L^1_{\omega_T}$.

**The generator as a quasimultiplier on $\mathcal{A}$**

The infinitesimal generator $\Delta_T$ of $T$ on $\mathcal{A}$ is the quasimultiplier on $\mathcal{A}$ defined by the formula

$$\Delta_T = e^{-i\alpha S_{\phi_T(f'_0)u/\phi_T(f_0)u}},$$

where $f_0 \in C^1([0, +\infty)) \cap \Omega \left( L^1_{\omega_T} \right)$ satisfies $f_0 = 0$, $f'_0 \in L^1_{\omega_T}$, and where $u \in \Omega(\mathcal{A})$. 
Let $B$ is a normalization of $A$ with respect to the semigroup $T$. Set $	ilde{\omega}_T = \|T(t)\|_{\mathcal{M}(B)}$, and set $\tilde{\phi}_T(f) = \int_0^{+\infty} f(t) T(te^{i\alpha})dt$ for $f \in L^1_{\tilde{\omega}_T}$, where the integral is computed with respect to the strong operator topology on $\mathcal{M}(B)$.

Generators and normalization of a Banach algebra with respect to the semigroup

Proposition: Let $B$ be a normalization of the Banach algebra $A$ with respect to the semigroup $T = (T(te^{i\alpha})_{t \geq 0}$, let $\lambda > \limsup_{t \to \infty} \frac{\log \|T(te^{i\alpha})\|}{t}$, and set $v_\lambda(t) = te^{-\lambda t}$ for $t \geq 0$. Then

$$\Delta_T = e^{-i\alpha} S\tilde{\phi}_T(v'_\lambda)u/\tilde{\phi}_T(v_\lambda)u \in Q\mathcal{M}(B) = Q\mathcal{M}(A),$$

where $u \in \Omega(B)$.

For holomorphic semigroups we have the following approach to the generator, already given in [3],

Definition: Let $T = (T(\zeta)_{\zeta \in S_{a,b}}$ be a holomorphic semigroup of multipliers on $A$. Set, for $\zeta \in S_{a,b}$ and $u \in \Omega(A),$

$$\Delta_{T,A} = ST'(\zeta)u/T(\zeta)u.$$
This definition of the generator of a holomorphic semigroup does not depend on the choice of $\zeta$ and $u$.

**Links between these notions of generators**

Proposition: Let $T = (T(\zeta)_{\zeta \in S_{a,b}}$ be a holomorphic semigroup of multipliers on $A$. For $\zeta \in S_{a,b}$, set $T_\zeta := (T(t\zeta))_{t>0}$, denote by $\Delta_{T,A}$ the generator of the holomorphic semigroup $T$ and denote by $\Delta_{T_\zeta,A}$ the generator of the semigroup $T_\zeta$, interpreted as quasimultipliers on $A$. Then we have

$$\Delta_{T,A} = \zeta \Delta_{T_\zeta,A}.$$ 

**Generator of the product of two commuting semigroups**

Proposition: Let $T_1 = (T_1(t))_{t>0}$ and $T_2 = (T_2(t))_{t>0}$ be two strongly continuous semigroups of multipliers on $A$. If $T_1(t)T_2(t) = T_2(t)T_1(t)$ for $t > 0$, then $T_1(s)T_2(t) = T_2(t)T_1(s)$ for $s > 0$, $t > 0$, $T_1 T_2 := (T_1(t)T_2(t))_{t>0}$ is a strongly continuous semigroup of multipliers on $A$, and we have

$$\Delta_{T_1T_2,A} = \Delta_{T_1,A} + \Delta_{T_2,A}.$$ 

A similar property holds for commuting semigroups holomorphic on the same sector.
Denote by $\hat{I}_T$ the character space of the Arveson ideal $I_T$, equipped with the usual Gelfand topology induced by the weak-* topology on the unit ball of the dual space of $I_T$. If $\chi \in \hat{I}_T$, then there exists a unique character $\tilde{\chi}$ on $QM(I_T)$ such that $\tilde{\chi}|_{I_T} = \chi$, defined by the formula $\tilde{\chi}(S_{u/v}) = \frac{\chi(u)}{\chi(v)}$ for $u \in I_T, v \in \Omega(I_T)$.

**Arveson spectrum of elements of $QM(I_T)$.**

**Definition:** The Arveson spectrum of $S = S_{u/v} \in QM(I_T)$ is the set

$$\sigma_{ar}(S) := \{\tilde{\chi}(S) : \chi \in \hat{I}_T\} = \{\chi(u)/\chi(v) : \chi \in \hat{I}_T\}.$$  

**Proposition:** The map $\chi \rightarrow \tilde{\chi}(\Delta_T, I_T)$ is a homeomorphism from $\hat{I}_T$ onto $\sigma_{ar}(\Delta_T, I_T)$.

Notice that the set $\{z \in \sigma_{ar}(\Delta_T, I_T) \mid \text{Re}(ze^{i\alpha}) \geq \lambda\}$ is compact for every $\lambda \in \mathbb{R}$ if $T = T(te^{i\alpha})_{t>0}$ is a strongly continuous semigroup of multipliers on $A$.

A character $\eta$ on $A_T$ satisfies $\eta(T(t)) = \tilde{\chi}(T(t))$ for some $\chi \in \hat{I}_T$ if and only if the map $t \rightarrow \eta(T(t))$ is continuous on $(0, +\infty)$, and in this situation

$$(\chi \circ \phi_T)(f) = \int_0^{+\infty} \eta(T(t))f(t)dt \text{ for } f \in L^1_{\omega_T}.$$
The resolvent takes values in the multiplier algebra of a normalization of $\mathcal{A}$.

In the following result, $\mathcal{B}$ denotes a normalization of the Banach algebra $\mathcal{A}$ with respect to the semigroup $T$, and $\Delta_T = \Delta_{T,A} \in \mathcal{QM}(\mathcal{A}) = \mathcal{QM}(\mathcal{B})$.

**Resolvent formulae**

**Proposition:** (i) Let $T = (Te^{i\alpha})_{t>0}$ be a strongly continuous semigroup of multipliers on $\mathcal{A}$. Then $\Delta_T - \lambda I$ is invertible in $\mathcal{QM}(\mathcal{A})$, $(\Delta_T - \lambda I)^{-1} \in \mathcal{M}(\mathcal{B}) \subset \mathcal{QM}_r(\mathcal{B}) = \mathcal{QM}_r(\mathcal{A})$ for $\lambda \in \mathbb{C} \setminus \sigma_{ar}(\Delta_T)$, the map $\lambda \to (\Delta_T - \lambda I)^{-1}$ is an holomorphic map from $\mathbb{C} \setminus \sigma_{ar}(\Delta_T)$ into $\mathcal{M}(\mathcal{B})$, and we have, for $Re(\lambda e^{i\alpha}) > \lim_{t \to +\infty} \frac{\log \|T(te^{i\alpha})\|}{t}$, $u \in \mathcal{B}$,

$$(\Delta_T - \lambda I)^{-1} u = - \int_0^{e^{i\alpha}\infty} e^{-\lambda s} T(s)uds.$$

(ii) Let $T = (T(\zeta))_{\zeta \in S_{a,b}}$ be a holomorphic semigroup of multipliers on $\mathcal{A}$. Then $\Delta_T - \lambda I$ is invertible in $\mathcal{QM}(\mathcal{A})$, $(\Delta_T - \lambda I)^{-1} \in \mathcal{M}(\mathcal{B}) \subset \mathcal{QM}_r(\mathcal{B}) = \mathcal{QM}_r(\mathcal{A})$ for $\lambda \in \mathbb{C} \setminus \sigma_{ar}(\Delta_T)$, the map $\lambda \to (\Delta_T - \lambda I)^{-1}$ is an holomorphic map from $\mathbb{C} \setminus \sigma_{ar}(\Delta_T)$ into $\mathcal{M}(\mathcal{B})$, and we have, for $Re(\lambda \zeta) > \lim_{t \to +\infty} \frac{\log \|T(t\zeta)\|}{t}$, $u \in \mathcal{B}$,

$$(\Delta_T - \lambda I)^{-1} u = - \int_0^{\zeta,\infty} e^{-\lambda s} T(s)uds.$$
Set $z\zeta = (z_1\zeta_1 + \cdots + z_k\zeta_k)$ and $e_z(\zeta) = e^{z\zeta}$ for $z = (z_1, \ldots, z_k) \in \mathbb{C}^k$, $\zeta = (\zeta_1, \ldots, \zeta_k) \in \mathbb{C}^k$.

Let $a = (a_1, \ldots, a_k)$, $b = (b_1, \ldots, b_k)$, with $a_j \leq b_j \leq a_j + \pi$, and denote by $M(a, b)$ the set of families $(\alpha, \beta) = (\alpha_1, \beta_1), \ldots, (\alpha_k, \beta_k)$ such that $\alpha_j = \beta_j = a_j$ if $a_j = b_j$ and such that $a_j < \alpha_j \leq \beta_j < b_j$ if $a_j < b_j$. Let $T = (T_1, \ldots, T_k)$ be a family of semigroups such that $T_j$ belongs to the class (1) with respect to $a_j$ if $a_j = b_j$ and such that $T_j$ belongs to the class (2) with respect to $a_j$ and $b_j$ if $a_j < b_j$. Let $W$ be the algebra of continuous functions $f$ on $\bigcup_{(\alpha, \beta) \in M(a, b)} \prod_{j \leq k} S_{\alpha_j, \beta_j}$, with the obvious convention if $\alpha_j = \beta_j$, such that $e_z(\zeta) f(\zeta) \to 0$ as $|\zeta| \to +\infty$ in $S_{\alpha, \beta} := \prod_{j \leq k} S_{\alpha_j, \beta_j}$ for every $z = (z_1, \ldots, z_k) \in \mathbb{C}^k$ and every $(\alpha, \beta) \in M(a, b)$, and such that the map $\zeta \to f(\zeta_1, \zeta_{j-1}, \zeta_j, \zeta_{j+1}, \ldots, \zeta_k)$ is holomorphic on $S_{a_j, b_j}$ for $(\zeta_1, \zeta_{j-1}, \zeta_{j+1}, \zeta_k) \in \prod_{l \neq j} \overline{S}_{\alpha_j, \beta_j}$ if $a_j < b_j$. For every element $\phi$ of the dual space $W'$ there exists $(\alpha, \beta) \in M(a, b)$, $z \in \mathbb{C}^k$ and a measure $\nu$ of bounded variation on $\overline{S}_{\alpha, \beta} := \prod_{j \leq k} \overline{S}_{\alpha_j, \beta_j}$ such that

$$< f, \phi > = \int_{\overline{S}_{\alpha, \beta}} e^{-z\zeta} f(\zeta) d\nu(\zeta) \quad (f \in W).$$
Let $\lambda = (\lambda_1, \ldots, \lambda_k) \in \mathbb{C}^k$ be such that $\lambda_j S_{\alpha_j, \beta_j} \subset S_{\alpha_j, \beta_j}$ if $\alpha_j < \beta_j$ and such that $\lambda_j > 0$ if $\alpha_j = \beta_j$. Set $T_\lambda(\zeta) = T_1(\lambda_1 \zeta_1) \cdots T_k(\lambda_k \zeta_k)$ for $\zeta = (\zeta_1, \ldots, \zeta_k) \in S_{\alpha, \beta}$.

If $\limsup_{|\zeta| \to +\infty} |e^{-z\zeta}||T_\lambda(\zeta)|| = 0$, it is possible to define the action of $\phi$ on $T_\lambda$ by using the formula

$$\langle T_\lambda, \phi \rangle u = \int_{S_{\alpha, \beta}} e^{-z\zeta} T_\lambda(\zeta) u d\nu(\zeta) \quad (u \in B).$$

Here the algebra $B$ is obtained by iterating the process of normalization of the algebra $A$ with respect to the semigroups $T_1, \ldots, T_k$, so that $B$ is a normalization of $A$ with respect to each of the semigroups $T_1, \ldots, T_k$.

Then $\langle T, \phi \rangle \in M(B) \subset QM_r(A)$. The convolution product of two elements of $W'$ associated to the same $(\alpha, \beta)$ may be defined in a natural way, and if $\lambda$ satisfies the conditions above we have

$$\langle T_\lambda, \phi_1 * \phi_2 \rangle = \langle T_\lambda, \phi_1 \rangle \langle T_\lambda, \phi_2 \rangle.$$

The usual Fourier-Borel transform of $\phi \in W'$ is defined by the formula

$$\mathcal{FB}(\phi)(z) = \langle e^{-z}, \phi \rangle$$

when $z$ is such that the first formula is satisfied with respect to some $(\alpha, \beta) \in M_{a,b}$ and some measure $\nu$ of bounded variation on $S_{\alpha, \beta}$ for every $f \in W$. 

There is a natural way to define the Fourier-Borel transform of $e_z T(\lambda)$ which takes values in $\mathcal{M}(\mathcal{B})$ and extends analytically to $-\text{Res}_{ar}(\Delta T(\lambda)) := \prod_{1 \leq j \leq k} \text{Res}_{ar}(-\lambda_j \Delta_{T_j})$, where $\text{Res}_{ar}(-\lambda_j \Delta_{T_j}) := \mathbb{C} \setminus -\lambda_j \sigma_{ar}(\Delta_{T_j})$:

$$\mathcal{FB}(e_z T(\lambda))(\zeta) = R(-\lambda \Delta_T, z + \zeta)$$

$$:= (-1)^k \prod_{1 \leq j \leq k} (\lambda_1 \Delta_1 + (z_1 + \xi_1)I)^{-1} \cdots (\lambda_k \Delta_k + (z_k + \xi_k)I)^{-1}.$$

Set $\overline{S}_{\alpha_j,\beta_j}^* = \overline{S}_{-\pi/2-\alpha_j,\pi/2-\beta_j}, \overline{S}_{\alpha,\beta}^* := \prod_{1 \leq j \leq k} \overline{S}_{\alpha_j,\beta_j}^*$.

**Action of linear functionals and Fourier-Borel transforms**

**Theorem:** We have, with respect to the strong operator topology on $\mathcal{B}$, denoting by $\delta_\eta$ the Dirac measure at $\eta$ considered as an element of $\mathcal{W}'$

$$< T(\lambda), \phi > = \lim_{\eta \to 0, \eta \in \mathcal{S}_{\alpha,\beta}} < e_{-\epsilon} T(\lambda), \phi * \delta_\eta >$$

$$= \lim_{\eta \to 0, \eta \in \mathcal{S}_{\alpha,\beta}} \lim_{\epsilon \to 0, \epsilon \in \mathcal{S}_{\alpha,\beta}} \frac{1}{2i\pi)^k} \int_{z+\partial \overline{S}_{\alpha,\beta}^*} \epsilon^{\eta(z-\sigma)} \mathcal{FB}(\phi)(\sigma) R(-\lambda \Delta_T, \sigma - \epsilon) d\sigma,$$

where $\partial \overline{S}_{\alpha,\beta}^* := \prod_{1 \leq j \leq k} \partial \overline{S}_{\alpha_j,\beta_j}^*$ denotes the "distinguished boundary" of $\overline{S}_{\alpha,\beta}^*$, and where $\partial \overline{S}_{\alpha_j,\beta_j}^*$ is oriented from $-ie^{-i\alpha_j \infty}$ to $ie^{-i\beta_j \infty}$. 

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Holomorphic functional calculus for finite families of commuting semigroups
### Action of the Fourier-Borel transform of linear functionals

**Definition:** For $\phi \in W'$ acting on $T(\lambda)$, the action of the Fourier-Borel transform of $\phi$ on $-\lambda \Delta_T$ is defined by the formula

$$\mathcal{FB}(\phi)(-\lambda \Delta_T) := \mathcal{FB}(\phi)(-\lambda_1 \Delta_{T_1}, \ldots, -\lambda_k \Delta_{T_k}) := < T(\lambda), \phi > .$$  

(1)

We introduce a class $\mathcal{U}$ of "admissible open sets" $U$, with piecewise $C^1$-boundary, of the form $(z + \mathcal{S}_{\alpha, \beta}^*) \setminus K$, where $K$ is bounded and where $(\alpha, \beta) \in \mathcal{M}_{a,b}$.

These open sets $U$ have the property that $U + \epsilon \subset U$ for $\epsilon \in \overline{\mathcal{S}_{\alpha, \beta}^*}$ and that $\overline{U} + \epsilon \subset \text{Res}_{ar}(-\lambda \Delta_T)$ for some $\epsilon \in \overline{\mathcal{S}_{\alpha, \beta}^*}$. Also $\Pi_{j=1}^k (-\lambda \Delta_{T_j} - I)^1$ is bounded on the distinguished boundary of $U + \epsilon$ for $\epsilon \in \mathcal{S}_{\alpha, \beta}^*$ when $|\epsilon|$ is sufficiently small.

### The class $H^{(1)}(U)$

**Definition:** When $U$ is admissible, $H^{(1)}(U)$ denotes the class of all holomorphic functions $F$ on $U$ such that

$$\|F\|_{H^{(1)}(U)} := \sup_{\epsilon \in \mathcal{S}_{\alpha, \beta}^*} \int_{\epsilon + \partial U} \|F(\sigma)\| \, |d\sigma| < +\infty$$

When $a_j = b_j$ for $j \leq k$, this space is the usual Hardy space $H^1$ on a product of open half-planes.
When an open set $U \subset \mathbb{C}^k$ is admissible with respect to $(\alpha, \beta) \in M_{a,b}$ and satisfies some more suitable admissibility conditions with respect to $T = (T_1, \ldots, T_k)$ and $\lambda \in \bigcup_{(\gamma, \delta) \in M_{a-\alpha, b-\beta}} S_{\gamma, \delta}$, a quasimultiplier $F(-\lambda_1 \Delta_{T_1}, \ldots, -\lambda_k \Delta_{T_k}) \in \mathcal{M}(B) \subset Q \mathcal{M}_r(A)$ is defined for $F \in H^{(1)}(U)$ by using the formula

$$F(-\lambda \Delta_T) := \frac{1}{(2\pi i)^k} \int_{\tilde{\partial} U} F(\zeta_1, \ldots, \zeta_k) \prod_{j=1}^k (\lambda_j \Delta_{T_j} + \zeta_j I)^{-1} d\zeta_1 \ldots d\zeta_k,$$

where $\tilde{\partial} U$ denotes the distinguished boundary of $U$, and where $\epsilon \in S^*_{\alpha, \beta}$ is chosen so that $\epsilon + U$ still satisfies the required admissibility conditions with respect to $T$ and $\lambda$. 

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Given $T$ and $\lambda \in \bigcup_{(\alpha,\beta) \in M_{a,b}} \overline{S}_{a-\alpha,b-\beta}$, denote by $\mathcal{W}_{T,\lambda}$ the family of all open sets $U \subset \mathbb{C}^k$ satisfying these admissibility conditions with respect $T$ and $\lambda$. Then $\mathcal{W}_{T,\lambda}$ is stable under finite intersections, $\bigcup_{U \in \mathcal{W}_{T,\lambda}} H^{(1)}(U)$ is stable under products and we have, for $F_1, F_2 \in \bigcup_{U \in \mathcal{W}_{T,\lambda}} H^{(1)}(U)$,

$$(F_1 F_2)(-\lambda_1 \Delta_{T_1}, \ldots, -\lambda_k \Delta_{T_k}) = F_1(-\lambda_1 \Delta_{T_1}, \ldots, -\lambda_k \Delta_{T_k}) F_2(-\lambda_1 \Delta_{T_1}, \ldots, -\lambda_k \Delta_{T_k}).$$
Action of elements of $H^\infty(U)$

For every $U \in \mathcal{W}_{T,\lambda}$ there exists $G \in H^{(1)}(U) \cap H^\infty(U)$ such that $G(-\lambda_1 \Delta T_1, \ldots, -\lambda_k \Delta T_k)(B)$ is dense in $B$, and for every $F \in H^\infty(U)$ there exists a unique $R_F \in \mathcal{QM}_r(B) = \mathcal{QM}_r(A)$ satisfying

$$R_F G(-\lambda_1 T_1, \ldots, -\lambda_k T_k) = (FG)(-\lambda_1 T_1, \ldots, -\lambda_k T_k) \quad (G \in H^{(1)}(U)).$$

The definition of $R_F$ does not depend on the choice of $U$. This gives the following result

**Theorem**

There exists a bounded algebra homomorphism from $F \rightarrow F(-\lambda_1 \Delta T_1, \ldots, -\lambda_k \Delta T_k)$ from $\bigcup_{U \in \mathcal{W}_{T,\lambda}} H^\infty(U)$ into $\mathcal{QM}_r(B) = \mathcal{QM}_r(A)$ such that $F(-\lambda_1 \Delta T_1, \ldots, -\lambda_k \Delta T_k)$ agrees with the previous definition if $F \in \bigcup_{U \in \mathcal{W}_{T,\lambda}} H^{(1)}(U)$, and $F(-\lambda_1 \Delta T_1, \ldots, -\lambda_k \Delta T_k) = T(\nu \lambda_j)$ if $F(\zeta) = e^{-\nu \zeta_j}$, where $\nu \lambda_j$ is in the domain of definition of $T_j$. 
Let $\phi \in W'$ acting on $T_\lambda$, associated to $(\alpha, \beta) \in M_{a,b}$. In general, the Fourier-Borel transform $FB(\phi)$ does not belong to $\bigcup_{U \in W_{T,\lambda}} H_\infty(U)$, so the $H_\infty$-functional calculus does not apply directly to $FB(\phi)$. On the other hand if we set $FB(\phi)_{\epsilon}(\zeta) = FB(\phi)(\epsilon + \zeta)$ for $\epsilon \in S_{\alpha,\beta}^*$, then $FB(\phi)_{\epsilon} \in \bigcup_{U \in W_{T,\lambda}} H_\infty(U)$, and we have, for $u \in B$, writing in red the formula obtained by using (1) and in blue the formulae obtained by using the $H_\infty$-functional calculus

$$FB(\phi)(-\lambda_1 \Delta T_1, \ldots, -\lambda_k \Delta T_k)u = \lim_{\epsilon \to 0} FB(\phi_{\epsilon})(-\lambda_1 \Delta T_1, \ldots, -\lambda_k \Delta T_k)u$$

$$= \lim_{\epsilon \to 0} FB(\phi)(\epsilon_1 I - \lambda_1 \Delta T_1, \ldots, \epsilon_k I - \lambda_k \Delta T_k)u \quad (u \in B).$$
Strongly outer functions on an open subset of $\mathbb{C}^k$

A function $F \in H^\infty(U)$ will be said to be strongly outer if there exists a sequence $(F_n)_{n \geq 1}$ of invertible elements of $H^{(\infty)}(U)$ such that $|F(\zeta)| \leq |F_n(\zeta)|$ and $\lim_{n \to +\infty} F(\zeta)F_n^{-1}(\zeta) = 1$ for $\zeta \in U$.

If $U$ is admissible with respect to some $(\alpha, \beta) \in M_{a,b}$ then there is a conformal map $\theta$ from $\mathbb{D}^k$ onto $U$ and the map $F \to F \circ \theta$ is a bijection from the set of strongly outer bounded functions on $U$ onto the set of strongly outer bounded functions on $\mathbb{D}^k$. Every bounded outer function on the open unit disc $\mathbb{D}$ is strongly outer, but the class of strongly outer bounded functions on $\mathbb{D}^k$ is smaller than the usual class of bounded outer functions on $\mathbb{D}^k$ if $k \geq 2$.

Strongly outer functions and dense principal ideals of $\mathcal{B}$

If $U \in \mathcal{W}_{T,\lambda}$, and if $F \in H^\infty(U)$ is strongly outer on $U$, then there exists $u \in \Omega(\mathcal{B}) \cap \text{Dom}(F(-\lambda_1 \Delta T_1, \ldots, -\lambda_k \Delta T_k))$ such that

$$F(-\lambda_1 \Delta T_1, \ldots, -\lambda_k \Delta T_k)u \in \Omega(\mathcal{B}).$$
If $U$ is an open subset of $\mathbb{C}^k$, the Smirnov class $S(U)$ is the class of holomorphic functions $F$ on $U$ such that there exists an outer function $G \in H^\infty(U)$ for which $FG \in H^\infty(U)$.

For every $U \in \mathcal{W}_{T,\lambda}$ and every $F \in S(U)$ there exists a unique quasimultiplier $R_F \in \mathcal{QM}(\mathcal{B}) = \mathcal{QM}(\mathcal{A})$ satisfying

$$R_F G(-\lambda_1 T_1, \ldots, -\lambda_k T_k) = (FG)(-\lambda_1 T_1, \ldots, -\lambda_k T_k)$$

for every $G \in H^\infty(U)$ such that $FG \in H^\infty(U)$. The definition of $R_F$ does not depend on the choice of $U$. This gives the following result:

**Theorem**

The algebra homomorphism $F \mapsto F(-\lambda_1 T_1, \ldots, -\lambda_k T_k)$ from $\bigcup_{U \in \mathcal{W}_{T,\lambda}} H^\infty(U)$ into $\mathcal{QM}_r(\mathcal{B}) = \mathcal{QM}_r(\mathcal{A})$ extends to an algebra homomorphism $F \mapsto F(-\lambda_1 T_1, \ldots, -\lambda_k T_k)$ from $\bigcup_{U \in \mathcal{W}_{T,\lambda}} S(U)$ into $\mathcal{QM}(\mathcal{B}) = \mathcal{QM}(\mathcal{A})$.

Moreover if $\chi$ is a character on $\mathcal{A}$, and if $\tilde{\chi}$ denotes the unique extension of $\chi$ as a character on $\mathcal{QM}(\mathcal{A})$, we have

$$\tilde{\chi}(F(-\lambda_1 T_1, \ldots, -\lambda_k T_k)) = F(-\lambda_1 \tilde{\chi}(\Delta T_1), \ldots, -\lambda_k \tilde{\chi}(\Delta T_k)) \quad (F \in \bigcup_{U \in \mathcal{W}_{T,\lambda}} S(U)).$$

Also if $F(\zeta_1, \ldots, \zeta_k) = -\zeta_j$ then $F(-\lambda_1 \Delta_1, \ldots, -\lambda_k \Delta_k) = \lambda_j \Delta T_j$. 

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https://hal.archives-ouvertes.fr/hal-01966621


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