Representations of Banach lattice algebras

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References


Banach algebras

An algebra is a linear, associative algebra over a field that is either $\mathbb{C}$ or $\mathbb{R}$. A Banach algebra is complex or real.

For example, consider the following:

(1) $C_0(K)$ or $C_0(K,\mathbb{R})$ for a locally compact space $K$, with uniform norm $| \cdot |_K$;

(2) $L^1(G)$ or $L^1(G,\mathbb{R})$ for a locally compact group $G$;

(3) $M(G)$ or $M(G,\mathbb{R})$ for a locally compact group $G$;

(4) $\mathcal{B}(E)$ for a (complex or real) Banach space $E$;

(5) $\ell^1(S,\omega)$, where $S$ is a semigroup and $\omega$ is a weight on $S$.

Given a real Banach algebra, there is a standard complexification; see Rickart.
Vector lattices

A real linear space $E$ is **partially ordered** if there is a partial order $\leq$ on $E$ such that $x+z \geq y+z$ when $x \geq y$ in $E$ and $z \in E$ and also $\alpha x \geq 0$ when $x \geq 0$ in $E$ and $\alpha \geq 0$ in $\mathbb{R}$. Set

$$E^+ = \{x \in E : x \geq 0\}.$$

The space $E$ is a **vector lattice** if, further, any two elements $x, y \in E$ have a sup called $x \vee y$ and an inf called $x \wedge y$, and then we have $|x| = x \vee (-x)$, $x^+ = x \vee 0$, and $x^- = (-x) \vee 0$, so that

$$x = x^+ - x^-, \quad |x| = x^+ + x^-.$$

The elements are **disjoint** if $|x| \wedge |y| = 0$, so that $x \perp y$. In this case, $|x + y| = |x| + |y|$.

Let $E$ be a vector lattice, with a subspace $F$. Then $F$ is a **sublattice** if $x \vee y \in F$ when $x, y \in F$, and $F$ is an **order ideal** if, further, $x \in F$ when $x \in E$, $y \in F$, and $|x| \leq |y|$.
Examples

A vector lattice is **order complete** (or **Dedekind complete**) if every non-empty subset that is bounded above has a supremum. Each vector lattice has an order completion.

(1) Take $K$ compact. The space $C(K, \mathbb{R})$ is a vector lattice. It is order complete iff $K$ is extremely disconnected; see [DDLS]. For example, $C(\beta \mathbb{N}, \mathbb{R})$ is order complete.

(2) Let $(X, \mu)$ be a measure space ($\mu$ has values in $[0, \infty]$), and take $p$ with $1 \leq p < \infty$. Then $L^p(X, \mu, \mathbb{R})$ is an order complete vector lattice – and also for $p = \infty$ when $\mu$ is $\sigma$-finite.

(3) Take a $\sigma$-algebra of subsets of a set $X$, and let $M(X, \mathbb{R})$ be the linear space of all signed measures $\mu : X \to \mathbb{R}$. This is an order complete vector lattice. For $\mu \in M(X, \mathbb{R})$, the usual total variation measure is $|\mu|$. 
Side remark

The following is deducible from [DDLS].

Let $K$ be a non-empty compact space. Then the Dedekind completion of $C(K)$ is $C(G_K)$, where $G_K$ is the Gleason cover of $K$. It is also equal to the Stone space of the complete Boolean algebra, $RO(K)$, which is the regular-open algebra of $K$.

There are lots of interesting properties of $G_K$. 
Regular maps

Let $E$ and $F$ be linear spaces. Then $L(E, F)$ is the space of all linear maps from $E$ to $F$.

Let $E$ and $F$ be vector lattices. A subset of $E$ is order bounded if it is contained in an interval $\{x \in E : a \leq x \leq b\}$ for some $a, b, \in E$. A linear map $T : E \to F$ is order bounded if it maps order bounded sets into order bounded sets. The set of these is the linear subspace $L_b(E, F)$ of $L(E, F)$.

Take $S, T \in L(E, F)$. Then $S \geq T$ if $Sx \geq Tx$ ($x \in E$). This gives a partial order on $L_b(E, F)$. The regular operators are elements of the subspace $L_r(E, F')$ of $L_b(E, F')$ that is spanned by the positive maps, so each regular operator has the form $S - T$, where $S$ and $T$ are positive.
Let $E$ and $F$ be vector lattices. In general, $L_b(E,F)$ and $L_r(E,F)$ are not vector lattices, but we have:

**Theorem** Suppose that $F$ is order complete. Then $L_b(E,F) = L_r(E,F)$ is also an order complete vector lattice, and we have the **Riesz–Kantorovitch formulae**, such as

$$|T|(x) = \sup\{|Ty| : |y| \leq x\}$$

and

$$(S \vee T)(x) = \sup\{Sy + Tz : y, z \in E^+, y + z = x\}$$

for $x \in E^+$. $\square$
Lattice homomorphisms

Let $E$ and $F$ be vector lattices. A linear map $T : E \to F$ is a lattice homomorphism if $T(x \wedge y) = Tx \wedge Ty$ for $x, y \in E$. Equivalently, we require that $|T(x)| = T(|x|)$ for $x \in E$.

These are positive, hence regular, linear operators.

The order dual space

Apply the above with $F = \mathbb{R}$. Then order bounded functionals are regular, and we obtain another vector lattice called $E^\sim$; it is the order dual space (and it is order complete).

The order adjoint of $T : E \to F$ is $T^\sim : F^\sim \to E^\sim$ given by $\langle T^\sim \varphi, x \rangle = \langle \varphi, Tx \rangle$ for $x \in E$ and $\varphi \in F^\sim$. 
Banach lattices

A Banach lattice is a (real) Banach space \((E, \| \cdot \|)\) such that \(\|x\| \leq \|y\|\) for \(x, y \in E\) with \(|x| \leq |y|\).

Examples Spaces \(C_0(K, \mathbb{R})\), \(L^p(X, \mu, \mathbb{R})\), and \(M(X, \mathbb{R})\) with \(\|\mu\| = |\mu|(X)\).

‘Automatic continuity’ : Let \(E\) and \(F\) be Banach lattices. Then an order bounded linear map \(T : E \to F\) is automatically continuous, and so we can write \(B_b(E, F)\) and \(B_r(E, F)\).

Advertisement: There are many connections with ‘multi-norms’: see [DLOT], for example.
Key facts

(1) Let $E$ be a Banach lattice. Then $E^\sim$ is the same as $E'$, the Banach space dual, and we obtain another Banach lattice.

(2) Let $E$ and $F$ be Banach lattices such that $F$ is order complete. Then $\mathcal{B}_r(E, F)$ is an order complete Banach lattice with respect to the regular norm $\| \cdot \|_r$, defined by

$$\| T \|_r = \| | T | \| \quad (T \in \mathcal{B}_r(E, F)).$$
Complex Banach lattices

Let $E$ be a (real) Banach lattice. Its complexified linear space $E_C$ has a modulus defined by

$$|x + iy|_C = \sup\{x \cos \theta + y \sin \theta : 0 \leq \theta \leq 2\pi\}$$

for $x, y \in E$. Then set $\|z\|_C = \| |z|_C \|$ ($z \in E_C$) to obtain a norm on $E_C$ giving a complex Banach space. This is a complex Banach lattice.

Examples

We obtain complex Banach lattices $C_0(K)$, $L^p(X, \mu)$, and $M(X)$ in the usual way (with the usual norms). □

Let $E_C$ and $F_C$ be such complex Banach lattices. An operator $T : E_C \rightarrow F_C$ has the form $S_1 + iS_2$, where $S_1, S_2 \in \mathcal{B}(E, F)$, and $T$ is order bounded or regular if both $S_1$ and $S_2$ have these properties. Further $T$ is a complex lattice homomorphism if $|Tz|_C = T(|z|_C)$ for $z \in E_C$. We have $\|T\|_{r,C} = \| |T|_C \|$, etc.
Banach lattice algebras

Definition Let $A$ be a (real) Banach lattice that is also a Banach algebra and such that $ab \in A^+$ whenever $a, b \in A^+$. Then $A$ is a Banach lattice algebra $= \text{BLA}$.

The theory of BLAs is somewhat rudimentary, but there are lots of examples, including in harmonic analysis. For a discussion, see [W2].

Examples $C_0(K, \mathbb{R}); B_r(E)$ for each order complete Banach lattice $E$.

Definition Let $A$ and $B$ be BLAs. A map $\pi : A \to B$ is a Banach lattice algebra homomorphism if it is an algebra homomorphism and a lattice homomorphism.

Such maps are automatically continuous.
Representations

**Definition** Let $E$ be an order complete Banach lattice. A **Banach lattice algebra representation** of a BLA $A$ on $E$ is BLA homomorphism $\pi : A \to \mathcal{B}_r(E)$.

**Definition** Let $A$ be a Banach algebra. Then the **left regular representation** of $A$ is the map $\pi : A \to \mathcal{B}(A)$ defined by

$$\pi(a)(b) = ab \quad (a, b \in A).$$

More generally, take $E$ to be a Banach left $A$-module and consider the representation $\pi : A \to \mathcal{B}(E)$ defined by $\pi(a)(x) = a \cdot x$ for $a \in A$ and $x \in E$.

In the case where $A$ is a BLA and $E$ is a Banach left $A$-module that is an order complete Banach lattice, we can consider whether the map $\pi : A \to \mathcal{B}_r(E)$ is a BLA homomorphism.
Order = Dedekind completions

Let $E$ be a Banach lattice. Then $E$ has an obvious Dedekind completion that is a Banach lattice. Now start from a BLA $A$. It would be nice to extend the product on $A$ to its Dedekind completion to form an order complete BLA, say $B$, and then look at BLA HMs from $A$ to $B_r(B)$.

Such an extension works in all the cases that we are interested in, but the general process seems to be somewhat murky.
Complex Banach lattice algebras

Let $A$ be a BLA. Then we have a complex Banach lattice $(A_{\mathbb{C}}, \| \cdot \|_{\mathbb{C}})$, and this is also a complex algebra for the obvious product.

In fact, $(A_{\mathbb{C}}, \| \cdot \|_{\mathbb{C}})$ is a complex Banach algebra, and we have

$$|ab| \leq |a| \cdot |b| \quad (a, b \in A_{\mathbb{C}}).$$

(Not quite trivial.)

We also have complex Banach lattice algebra homomorphisms and complex Banach lattice algebra representations.
Locally compact groups

Let $G$ be a locally compact group, with left Haar measure $m_G$. Then we have the order complete Banach lattice $M(G, \mathbb{R})$ which is isometrically lattice isomorphic to the dual Banach lattice $C_0(G, \mathbb{R})'$. With respect to convolution multiplication, $M(G, \mathbb{R})$ is a BLA. Further, the closed ideal $L^1(G, \mathbb{R})$ is also an order complete BLA.

The process of complexification gives us the usual $M(G)$ and $L^1(G)$; they are complex Banach lattice algebras.
Representations

The **basic question** that we consider is the following. The left regular representation \( \pi \) of \( M(G, \mathbb{R}) \) takes its values in the algebra of regular operators \( B_r(M(G, \mathbb{R})) \), and \( \pi \) is a continuous Banach algebra \( \text{HM} \). Since \( M(G, \mathbb{R}) \) and \( B_r(M(G, \mathbb{R})) \) are BLAs, it is natural to ask if \( \pi \) is also a Banach lattice homomorphism.

This is an open question in various lists, including [W3].

We shall show that more than this is true.

More generally of course: Is there a similar result for every order complete BLA \( A \)? What about regular homomorphisms \( A \to B_r(E) \) for suitable \( E \)?
An easy result

Let $K$ be an extremely disconnected compact space. Then we claim that the left regular HM of $C(K)$ is a lattice HM.

Define $L_g(h) = gh$ ($g, h \in C(K)$), and take $f \in C'(K)$. Then $L_{|f|}$ is an upper bound for $L_f$ and $-L_f = L_{-f}$.

Suppose that $T \in \mathcal{B}_r(C(K))$ is also an upper bound. Take $g \in C(K)^+$. For $x \in K$, we have

$$(Tg)(x) \geq (fg)(x)$$

and

$$(Tg)(x) \geq (L_{-f}g)(x) = -(fg)(x),$$

and so $(Tg)(x) \geq (|f|g)(x)$. Hence $Tg \geq L_{|f|}g$.

This shows that $T = L_{|f|}$, and so $|L_f| = L_{|f|}$.

**Theorem** Suppose that $K$ is extremely disconnected. Then the left regular representation $\pi : C(K) \to \mathcal{B}_r(C'(K))$ is a BLA homomorphism. \qed
A generalization

For general compact $K$, with Gleason cover $G_K$, the space $C(G_K)$ is a Banach $C(K)$-module and an order complete Banach lattice.

**Theorem** The left regular representation $\pi : C(K) \to B_r(C(G_K))$ is a BLA HM. $\square$
Earlier results

Results of Brainerd and Edwards (1963), Gilbert (1968), and Arendt (1981) show (in a rather complicated way):

Let $G$ be a locally compact group. Then the left representation $\pi : M(G) \to B_r(L^1(G))$ is an isometric embedding and a BLA homomorphism. There is a shorter proof in [GL], Theorem 3.1. This also true for the actions on $L^p(G)$ when $1 < p < \infty$ whenever $G$ is amenable.

Set $A = B_r(E')$, where $E$ is an order complete Banach lattice. Then the map $A \to B_r(A)$ is a Banach lattice homomorphism (Schep, 1989).

There are also strong partial results of Marcel and Kok; see [K]. For example, the representation of $L^1(G)$ on $B_r(L^p(G))$ is a BLA HM even when $G$ is not amenable, but this does not extend to a representation of $M(G)$ on $B_r(L^p(G))$ or $B_r(M(G))$; these remained open.
Locally compact spaces

Now $X$ is a locally compact space. For each $f \in C(X)$, its support is $\text{supp } f$, the closure of the set $\{x \in X : f(x) \neq 0\}$. For a non-empty $S$ in $X$, we have

$$C(X, \mathbb{R}, S) = \{f \in C(X) : \text{supp } f \subset S\},$$

and then

$$C_c(X, \mathbb{R}) = \bigcup C(X, \mathbb{R}, K)$$

for $K$ compact in $X$. The space $C_c(X, \mathbb{R})$ is a vector lattice. Further, there is a natural locally convex topology on $C_c(X, \mathbb{R})$: a linear map from $C_c(X, \mathbb{R})$ into a LCS is continuous iff its restriction to each $C(X, \mathbb{R}, K)$ is continuous. (It is an inductive limit topology.)
Radinon measures

A Radon measure on $X$ (in the sense of Bourbaki) is a continuous linear functional on $C_c(X, \mathbb{R})$; the space is called $\mathcal{M}(X)$ in Bourbaki.

We can identify the space $C_c(X, \mathbb{R})^\sim$.

**Fact** The order dual space $C_c(X, \mathbb{R})^\sim$ of the vector lattice $C_c(X, \mathbb{R})$ is just $\mathcal{M}(X)$.

We can tie in $\mathcal{M}(X)$ with ‘regular Borel measures’, but the terminology differs in the literature.
Supports

Let \( U \) be a non-empty, open subset of \( X \). An element \( \varphi \) of \( C_c(X, \mathbb{R}) \) vanishes on \( U \) if \( \langle \varphi, f \rangle = 0 \) whenever \( f \in C(X, \mathbb{R}, U) \). Let \( V \) be the union of the open sets in \( X \) on which \( \varphi \) vanishes. Then the support of \( \varphi \) is \( X \setminus V \); call it \( \text{supp} \varphi \).

The subset of elements \( \varphi \) of \( C_c(X, \mathbb{R}) \) with \( \text{supp} \varphi \subset S \) is an order ideal in \( C_c(X, \mathbb{R}) \).

Technical remarks: (1) Two elements of \( C_c(X, R) \) that have disjoint supports are disjoint elements of \( C_c(X, R) \).

(2) For a closed subspace \( Y \) of \( X \), there is an embedding of \( C_c(Y, \mathbb{R}) \) in \( C_c(X, \mathbb{R}) \). (Not quite trivial.)

Definition Take \( \varphi \in C_c(X, \mathbb{R}) \). Then \( \varphi \) has separated supports if \( \text{supp} \varphi^+ \) and \( \text{supp} \varphi^- \) are disjoint subsets of \( X \).

(We can have \( \text{supp} \varphi^+ = \text{supp} \varphi^- = X \).)
Embedding familiar vector lattices in $C_c(X, \mathbb{R})^\sim$

Let $\mu$ be a positive regular Borel measure on a locally compact space $X$, and suppose that $g : X \to \mathbb{R}$ is Borel measurable. Then $g$ is **locally integrable** with respect to $\mu$ if
\[
\int_K |g(x)| \, d\mu < \infty \quad \text{for each compact } K \subset X.
\]
The collection of these functions (usual identification) forms a vector lattice, called $L^{1,\text{loc}}(X, \mu, \mathbb{R})$, the **locally integrable functions**.

**Theorem** For such a $g \in L^{1,\text{loc}}(X, \mu, \mathbb{R})$, define $\varphi_g$ by
\[
\langle \varphi_g, f \rangle = \int_X f g \, d\mu \quad (f \in C_c(X, \mathbb{R})).
\]
Then the map $g \mapsto \varphi_g$ is an injective lattice HM from $L^{1,\text{loc}}(X, \mu, \mathbb{R})$ into $C_c(X, \mathbb{R})^\sim = \mathcal{M}(X)$. □
Embedding measures

**Theorem** Let $X$ be a locally compact space. For each $\mu \in M(X)$, set

$$\langle \varphi_\mu, f \rangle = \int_X f \, d\mu \quad (f \in C_c(X)).$$

Then $\varphi_\mu \in C_c(X)^\sim$, and the map

$$\mu \mapsto \varphi_\mu, \; M(X) \to C_c(X)^\sim,$$

is an injective lattice HM. Further, the set of $\varphi \in M(X)$ with compact, separated support is a dense subspace of the image.

It is not quite trivial that it is a lattice HM; the proof uses the Riesz–Kantorovich formulæ and Radon–Nikodym theorem. Then use the Riesz representation theorem to show that it is an injection.
Other examples

**Fact** Take $p$ with $1 \leq p < \infty$. Then $L^p(X, \mu, \mathbb{R})$ is a vector sublattice of $L^{1,\text{loc}}(X, \mu, \mathbb{R})$ and so $L^p(X, \mu, \mathbb{R})$ is identified with an order complete Banach lattice in $C_c(X, \mathbb{R})\sim$. Further, the functions in $L^p(X, \mu, \mathbb{R})$ with compact, separated support is a dense subspace of $L^p(X, \mu, \mathbb{R})$. □

**Fact** Let $G$ be a locally compact group, and let $S$ be a closed subspace that is a subsemigroup. Suppose that $\omega$ is a continuous weight on $S$. Then $M(S, \omega, \mathbb{R})$ is a vector sublattice of $L^{1,\text{loc}}(X, \mu, \mathbb{R})$ with similar properties. □
The main theorem

**Theorem** Let $G$ be a locally compact group, and let $X$, $Y$, and $Z$ be vector sublattices of $C_c(G, \mathbb{R})^\sim = \mathcal{M}(G, \mathbb{R})$, with $Z$ order complete. Suppose that $\star : X \times Y \to Z$ is a bilinear map such that $x \star y \in Z^+ \ (x \in X^+, y \in Y^+)$. Define $\pi : X \to L_r(Y, Z)$ by

$$\pi(x)(y) = x \star y \quad (x \in X, y \in Y),$$

and suppose that the following hold:

(1) $\text{supp} \ (x \star y) \subset \text{supp} \ x \cdot \text{supp} \ y$ for each $x \in X^+$ and $y \in Y^+$ with compact support;

(2) the elements in $X$ with compact, separated support are dense in $X$;

(3) the elements in $Y^+$ with compact support are dense in $Y^+$;

(4) $\chi_A y \in Y$ whenever $y \in Y^+$ has compact support and $A$ is a Borel subset of $\text{supp} \ y$.

Then $\pi$ is a lattice HM.
**Sketch of the proof**

We have to show that $|\pi(x)| = \pi(|x|)$ for each $x \in X$.

Since the maps $x \mapsto |\pi(x)|$ and $x \mapsto \pi(|x|)$ are both continuous, it is sufficient to prove this when $x \in X$ has separated, compact support. Fix such an $x = x^+ - x^-$. Then there is a relatively compact neighbourhood $U$ of $e_G$ with $(\text{supp} \: x^+)U \cap (\text{supp} \: x^-)U = \emptyset$.

We need $|\pi(x)|(y) = \pi(|x|)(y)$ for all $y \in Y$. It is sufficient to suppose that $y \in Y^+$ and that $y$ has compact support.
Sketch continued

First suppose, further, that supp \( y \subset U_s \) for some \( s \in G \). Certainly \( |\pi(x)|_b(y) \leq \pi(|x|)(y) \) because \( \pi \) is positive. Also \( |\pi(x)|_b(y) \geq |\pi(x)(y)| \). But we have supp \( (x^+ \ast y) \subset (\text{supp } x^+) \cdot U_s \) and supp \( (x^- \ast y) \subset (\text{supp } x^-) \cdot U_s \), and so

\[
|\pi(x)(y)| = |x^+ \ast y - x^- \ast y| = |x^+ \ast y| + |x^- \ast y| = \pi(|x|)(y).
\]

Thus \( |\pi(x)(y)| \geq \pi(|x|)(y) \). So we win in this special case.

Now suppose just that \( y \in Y^+ \) and that \( y \) has compact support. Choose an open neighbourhood \( V \) of \( e_G \) with \( \overline{V} \subset U \). Then supp \( y \) is contained in a union of finitely many right translates of \( V \). Then \( y \) is a finite sum of elements of \( Y^+ \) each of which is supported in a right translate of \( V \). So we win by linearity. \( \square \)
Consequences

**Theorem** Let $G$ be a locally compact group. Then the left regular representation
\[ \pi : M(G) \to \mathcal{B}_r(M(G)) \] is an isometric Banach lattice algebra homomorphism.

**Proof** Put $M(G, \mathbb{R})$ into $C_c(G, \mathbb{R})^\sim$ as earlier, with image $X$, and apply the theorem taking $Y = Z = X$ and $\star$ to be the usual convolution product. The complex case then follows. $\square$

Similarly, extending Arendt and Ghahramani–Lau.

**Theorem** Let $G$ be a locally compact group, and take $p$ with $1 \leq p < \infty$. Then the map
\[ \pi : M(G) \to \mathcal{B}_r(L^p(G)) \] where
\[ \pi(\mu)(g)(s) = \int_G g(t^{-1}s) \, d\mu(t) \quad (s \in G) \]
for $g \in L^p(G)$ and $\mu \in M(G)$, is an injective Banach lattice algebra homomorphism. $\square$
Further results

By variations of the above, we obtain:

**Theorem** Let $A$ be an order complete BLA. Then the left regular representation of $A$ is a Banach lattice algebra homomorphism from $A$ into $\mathcal{B}_r(A)$ in the following cases:

(1) $A = M(G)$ or $L^1(G)$ for a locally compact group $G$;

(2) $A = M(S, \omega)$, where $S$ is a closed semigroup of a locally compact group and $\omega$ is a continuous weight on $S$ (so that $A$ is a Beurling algebra);

(3) $A = \ell^1(S, \omega)$, where $S$ is a cancellative semigroup and $\omega$ is a weight on $S$ (such examples could be radical BAs);

(4) ‘Twisted Orlicz algebras’ of Öztop and Samei.
Counter-examples

On the other hand Tony Wickstead [W1] has shown that there are lots of finite-dimensional mutually non-isomorphic, commutative BLAs (with positive identity element) that have no faithful, finite-dimensional Banach lattice representations at all.

Some questions

Can we characterize the BLAs $A$ for which the previous theorem works? Maybe go for a representation of $A$ in $L_r(E)$ for a more general order complete Banach lattice $E$?

Does the above representation theorem lead to some development of the theory of Banach lattice algebras?