BERNOULLI NUMBERS, UMBRAL CALCULUS, AND UNIVERSAL ALGEBRA

BOB QUACKENBUSH

ABSTRACT. Define $B^0 := 1$ and $(B+1)^n - B^n = 0$ for $n \geq 2$. That is, $B_0 = 1$, $B_1 = -1/2$, $B_2 = 1/6$, $B_3 = 0$, $B_4 = 1/30$, $B_5 = 0, \ldots$. We refer to B_k as the k^{th} *Bernoulli number*. Let us rewrite this definition into one equation:

$$e^{(B+1)x} - e^{Bx} = x.$$

Hence,

$$x = e^{(B+1)x} - e^{Bx} = e^{Bx}e^x - e^{Bx}1 = e^{Bx}(e^x - 1),$$

and so,

$$e^{Bx} = \frac{x}{e^x - 1}.$$

But $\frac{x}{e^x-1} + \frac{x}{2}$ is an even function, and so $B_{2n+1} = 0$ for $n \geq 1$.

This is the basic Umbral Calculus argument for the Bernoulli numbers; I will explain Umbral Calculus in terms of modern algebra.