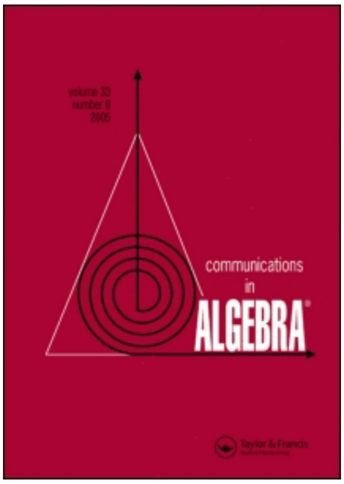

Communications in Algebra
Publication details, including instructions for authors and subscription information:
http://www.informaworld.com/smpp/title~content=t713597239

EXPLICIT DESCRIPTIONS OF THE INDECOMPOSABLE INJECTIVE MODULES OVER JATEGAONKAR'S RINGS

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Online publication date: 12 April 2002

To cite this Article Kucera, T. G.(2002) 'EXPLICIT DESCRIPTIONS OF THE INDECOMPOSABLE INJECTIVE MODULES OVER JATEGAONKAR'S RINGS', Communications in Algebra, 30: 12, 6023 — 6054
To link to this Article: DOI: 10.1081/AGB-120016030
URL: http://dx.doi.org/10.1081/AGB-120016030

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COMMUNICATIONS IN ALGEBRA®
Vol. 30, No. 12, pp. 6023–6054, 2002

EXPLICIT DESCRIPTIONS OF THE INDECOMPOSABLE INJECTIVE MODULES OVER JATEGAONKAR'S RINGS

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ABSTRACT

In 1968–1969, A. V. Jategaonkar published his famous constructions of left but not right noetherian rings that provided counterexamples to several important conjectures of that era. These examples, and others like them, seemed to indicate that, in general, the task of completely understanding the structure of indecomposable injective modules over one-sided noetherian rings was hopeless. In this paper I show how to deduce by natural methods, directly from the known description of these rings and their properties, explicit computational descriptions of the indecomposable injective left modules over Jategaonkar's rings. I use these explicit descriptions to answer some simple structural questions about the indecomposables.

DOI: 10.1081/AGB-120016030 0092-7872 (Print); 1532-4125 (Online)
 www.dekker.com

The structure of indecomposable injective modules over a commutative noetherian ring has been well understood since the work of Matlis[10] in the 1950's. There is a natural correspondence between the indecomposable injectives and the localizations of the ring, and the structure of the indecomposables is preserved by the localizations. An indecomposable injective is the union of an ascending chain (of order type at most ω) of extensions by finite dimensional vector spaces. In the case of non-commutative (one- or two-sided) noetherian rings, the connection between the structure of injective modules and the behaviour—even the possibility—of localization remains (see, e.g., Jategaonkar[4] for an extended argument as to why this is so), but very little is understood about how the indecomposable injectives are put together in general. Indeed, the book of Jategaonkar just cited provides ample evidence that indecomposable injectives over one-sided noetherian rings can have wildly varying kinds of structure, and that some examples may be, in some sense, inaccessible to our understanding. In remarking about the difficulties associated with one-sided noetherian rings, Jategaonkar[4, p. 91] says "Our decision to work with Noetherian rings rather than right Noetherian ones is often dictated by the exigencies of the situation under consideration. We note though that, after Jategaonkar (69) [that is[3], of the current bibliography], an attempt to stay with right Noetherian rings at all costs is generally regarded as futile." The family of examples developed in the paper cited are important and fascinating: they simultaneously provide counter-examples to a handful of different conjectures of the time, and they illustrate how strongly the left ideal structure of a ring can be disconnected from the right ideal structure of the ring. Nonetheless, they *do not* provide examples of indecomposable injectives that are difficult to understand or impossible to describe. It is the purpose of this paper not only to describe the indecomposable injectives over these badly behaved rings, but to convince the reader that the structure of the indecomposables is easily and naturally deduced from elementary facts about injective modules combined with the description of the rings themselves.

In actual fact, this latter task would take up far too much paper for the family of rings in[3]. Instead, I start with a much simplified version of these rings, the ring R treated in Example 3.3.8 of Jategaonkar's book.[4] This ring is just a homomorphic image of the first member of the more general family. After describing the ring, in Sec. 2 I take the reader step-by-step through a natural process (complete with one intentional mistaken initial guess!) that leads to explicit descriptions of the two indecomposable injectives over the ring R. The description is sufficiently explicit to make it routine to check, after the fact, that the structures that we have deduced actually are the indecomposable injectives for which we were looking. This is followed in Sec. 3 by showing how this explicit description can be used to solve certain kinds of problems related to the indecomposables. The process of showing how to deduce from elementary facts what the indecomposable injectives look like in the general version of the examples is not any more instructive than the shorter exposition presented for the simple version of the example. As a result, in the final section I content myself with describing the examples and verifying their correctness. This is in fact in itself a fairly lengthy task. I conclude the section by applying the explicit descriptions to analyze some of the internal structure of these indecomposable injectives.

The idea that such constructions might be possible was inspired in part by my work on some explicit descriptions of complicated indecomposable injective modules over certain commutative noetherian rings[6] and by example 9.3.7 in Jategaonkar's book,[4] where the same sort of construction is used to build an explicit description in a non-commutative noetherian case. The opportunity to read an early version of Musson's paper[11] also helped.

The approach taken here to injective modules is to view them as structures in which we can solve systems of linear equations. For more details on this approach and its history, and in particular on the equivalency with the usual definition, see my earlier paper.[6] Here I just repeat enough to ensure readability of this paper.

A *linear equation* (in variables $\bar{v}$) (over a right R-module M) is just an R-linear combination of the variables in $\bar{v}$ set equal to some constant from M. A *system* of linear equations in possibly infinitely many variables $\bar{v}$ over M is a set (again possibly infinite) of linear equations over M, each in finitely many variables from $\bar{v}$. A *solution* to such a system in some $N \supseteq M$ is just an assignment of values in N to each of the variables of $\bar{v}$ that makes each of the equations true in N. Note that of course a finite system of equations can be presented in matrix form as $\bar{v}A = \bar{m}$, where A is a matrix over R and $\bar{m}$ is a tuple in the right R-module M. Such a finite system of equations is *consistent* if for every matrix B over R (of the right shape) such that $AB = 0$, we also have $\bar{m}B = 0$. An infinite system of linear equations is *consistent* if every finite subset of it is consistent. It is straightforward to see that a system of equations over M is consistent if and only if it has a solution in some extension of M.

A right R-module E is *injective* if every consistent system of linear equations over E (possibly infinite, with infinitely many variables) has a solution in E itself. *Baer's criterion* for injectivity in this formulation reads: E is injective if every (possibly infinite) system of linear equations over E in *one* variable has a solution in E itself.

I remind the reader that the *injective envelope* $\mathrm{E}_R(M)$ of the right R-module M is characterized variously as a maximal essential extension of M, as a minimal injective extension of M, or as an injective essential extension of M. (In the language of linear equations, a module N is an essential extension of its submodule M if and only if every non-zero element of N satisfies a non-trivial linear equation over M.) The injective envelope of M always exists. An injective module is (direct sum) indecomposable if and only if it is the injective envelope of R/I for some meet-irreducible right ideal I.

Much of the work here has been motivated by my work in the model theory of modules. The main body of the material presented here requires no knowledge of mathematical logic or the model theory of modules, but some of the applications of these results are to problems in the model theory of modules (Sec. 3.1, 3.3).

Notation. The symbol $\langle\!\langle \cdots \rangle\!\rangle$ is used to denote the submodule (ideal, etc) generated by "$\cdots$". Scalar multiplications are indicated by a large dot: $m{\bullet}r$. Ordered pairs are enclosed in angle brackets: $\langle a, b\rangle$.

1 DESCRIPTIONS OF THE RINGS

I follow Jategaonkar's descriptions in the original sources as closely as possible. So in particular in the first example, I work "on the right", and in the second example I work "on the left".

1.1 The Ring *R*—A Simple Case

In the first main Section, I study the two injective modules over the ring R of example 3.3.8 in Jategaonkar.[4] This ring is, in essence, a much simplified version of the more complicated examples of Jategaonkar[3], which will be treated in the final section of this paper. There are a few alternative ways of presenting this ring: for instance, as a *trivial* extension (cf.[1]), or as an image of a twisted polynomial ring; but I will follow Jategaonkar's exposition closely.

Let k be a field and $F = k(x_n : n \geq 1)$, the rational function field in countably many commuting indeterminates over the field k. Let $D = F[x]_{\langle\!\langle x \rangle\!\rangle}$ and note that $F(x)$ is the quotient field of D. Let $\sigma : D \longrightarrow F$: $(x \mapsto x_1, x_n \mapsto x_{n+1}, (n \geq 1))$ be a k-algebra homomorphism. Note that σ is one-to-one but not onto, and is not nilpotent. Note also that σ extends naturally to a k-algebra isomorphism of $F(x)$ onto F. Let ρ be the F-algebra homomorphism $\rho : D \longrightarrow F : d \mapsto d(0)$, evaluation of d at 0 for x. Note that ρ is the identity on F. These two maps make F into a (D, D)-bimodule B with scalar actions $d{\bullet}b = \sigma(d)b$ and $b{\bullet}d = b\rho(d)$, (showing how to construe

R as a trivial extension). The ring R then has $B \oplus D$ as the underlying abelian group, with the multiplication determined by:

$$\langle b, d\rangle\langle b', d'\rangle = \langle b\rho(d') + \sigma(d)b', dd'\rangle.$$

The ring R is local, right noetherian, not left noetherian, and not right classical. It has subrings $F \subset D \subset R$, $(D \cong 0 \oplus D)$. The identity element is $\langle 0, 1\rangle$ and the units are all elements $\langle b, d\rangle$ with $d \notin xD$, in which case $\langle b, d\rangle^{-1} = \left\langle \frac{-b}{\rho(d)\sigma(d)}, \frac{1}{d}\right\rangle$. Thus $\mathrm{J}(R) = xR = B \oplus xD$, so $\langle b, d\rangle \in \mathrm{J}$ iff $\rho(d) = 0$. I will use x^n for $\langle 0, x^n\rangle$. The right zero-divisors are precisely the elements of J, the left zero-divisors are the elements of the form $\langle b, 0\rangle$, and if l is any left zero-divisor and r is any right zero-divisor, then $lr = 0$.

Every right ideal of R is principal and two-sided; no non-trivial two-sided ideal is finitely generated as a left ideal. These ideals are linearly ordered by inclusion:

$$R \supset J \supset J^2 \supset \cdots \supset J^\omega \supset 0$$

Note that as right ideals, $J^n = \langle\langle x^n\rangle\rangle$ and $J^\omega = \langle\langle\langle 1, 0\rangle\rangle\rangle = B \oplus 0$. Every right ideal is irreducible, and J and J^ω are the only prime ideals. Note that $J \cdot J^\omega = J^\omega$ whereas $J^\omega \cdot J = 0$. It follows from all this that there are just two indecomposable injective right R-modules: $\mathrm{E}(R/J)$ and $\mathrm{E}(R/J^\omega)$. (I will note in passing that this latter fact can also be given a straightforward model-theoretic explanation: The indecomposable injectives are in one-to-one correspondence with the non-orthogonality classes of strongly regular types.[5] Every strongly regular type is non-orthogonal to a type (in the imaginary universe T^{eq}) of rank ω^α for some α[9], and the biggest theory of injectives has complete elimination of imaginaries.[8] The two 1-types of rank ω^α are "$\mathrm{ann}(v) = J$", of rank $\omega^0 = 1$, and "$\mathrm{ann}(v) = J^\omega$" of rank $\omega^1 = \omega$.)

For proving some of the above statements, and for following the computations that will be presented in the next section, it is useful to have a catalogue of factorizations in R, as follows: If $d \neq 0$ then

$$\begin{aligned}\langle b, 0\rangle &= \langle 0, d\rangle\langle b, 0\rangle\langle 0, \sigma(d)^{-1}\rangle \\ &= \langle 0, d\rangle\langle b\sigma(d)^{-1}, 0\rangle \\ &= \langle 1, 0\rangle\langle 0, b\rangle\end{aligned}$$

If $\rho(d) \neq 0$ then

$$\langle b, d\rangle\langle b', d'\rangle = \left\langle \frac{b\rho(d') + \sigma(d)b' - \sigma(d')b}{\rho(d)}, d', \right\rangle\langle b, d\rangle$$

If $d \neq 0$ then

$$\langle b', d'\rangle\langle b, d\rangle = \langle b, d\rangle\left\langle\frac{b'\rho(d) + \sigma(d')b - b\rho(d')}{\sigma(d)}, d'\right\rangle$$

and

$$\langle 1, 0\rangle = \langle b, 0\rangle\langle 0, 1/b\rangle = \langle 0, d\rangle\langle\sigma(d)^{-1}, 0\rangle$$

In general (because the image of σ is fixed by ρ),

$$\langle 0, d\rangle\langle 1, 0\rangle = \langle 1, 0\rangle\langle 0, \sigma(d)\rangle$$

1.2 The Rings $\mathcal{R}^\alpha$ — The General Case

Now I describe the more complicated rings from[3]. It would be nice to follow Jategaonkar's description exactly, but unfortunately a choice that he makes in order to make the exposition of his construction simpler (starting the enumeration of indeterminates with x_1 rather than with x_0) would make the description of my constructions unwieldy, with separate cases for finite and infinite β. So with the minor change in the indexing of indeterminates, I present Jategaonkar's examples.

In[3, see especially Theorem 4.6], Jategaonkar shows the existence of rings R^α, α an ordinal, with the following description and properties.

There is a division ring $K \subset R^\alpha$ and twisted polynomial extensions of K inside R^α satisfying:

$$\begin{aligned} \bar{R}_\beta &= K[x_\gamma, \rho_\gamma : \gamma < \beta] \quad \text{for each} \quad \beta \leq \alpha. \\ R_\beta &= \bar{R}_\beta[x_\beta, \rho_\beta] \qquad \text{for each} \quad \beta < \alpha. \\ R^\alpha &= \bar{R}_\alpha\,. \end{aligned}$$

where each $\rho_\beta : \bar{R}_\beta \longrightarrow K$ is a monomorphism, and multiplication in R^α is determined by $x_\beta r = \rho_\beta(r)x_\beta$ for any r in $\bar{R}_\beta$.

Elements of R^α can be expressed in an essentially unique way as a (finite) sum of distinct *standard monomials*, where a standard monomial is a term of the form $a\,x_{\alpha_1}^{n_1}\cdots x_{\alpha_k}^{n_k}$ for some $k \geq 0$, $a \in K$, $\alpha_1 < \cdots < \alpha_k < \alpha$, and $n_i > 0$. (For $k = 0$, a standard monomial is just some element of K.) Jategaonkar shows that R^α is a principal left ideal domain and that the elements $1 + x_\beta$, $\beta < \alpha$, are right R^α-linearly independent. From these facts follow most of the interesting and peculiar properties of the ring R^α.

Let Θ be the set of all monic standard monomials (including 1). The set Θ can be ordered in order type ω^α as follows. The least element is 1, and otherwise given two monic standard monomials, write them with common variables as $\theta_1 = x_{\alpha_1}^{n_1} \cdots x_{\alpha_k}^{n_k}$ and $\theta_2 = x_{\alpha_1}^{m_1} \cdots x_{\alpha_k}^{m_k}$, $\alpha_1 < \cdots < \alpha_k < \alpha$ and $n_i \geq 0$, $m_i \geq 0$. Then $\theta_1 \prec \theta_2$ if and only if for some t, $n_l = m_l$ for all l, $t < l \leq k$, and $n_t < m_t$. Consequently, each standard monomial has a well defined degree which is an ordinal $< \omega^\alpha$, and I define the degree of a non-zero element of R^α to be the maximum of the degrees of its terms. Note in particular that $\deg(1) = 0$ and that $\deg(x_\beta) = \omega^\beta$. Note also that any subsequence of a monic standard polynomial is again a member of Θ, in particular initial and final segments (including the empty segment, taken to be 1) of some $\theta \in \Theta$ are again in Θ. For any $\theta \in \Theta$ and $a \in K$, I define the *order* of $a\theta$ to be -1 if $\theta = 1$, otherwise it is the largest index of a variable occurring in θ. For $\theta \in \Theta$, $\theta = x_{\alpha_1}^{n_1} \cdots x_{\alpha_k}^{n_k}$, I let $\rho_\theta = \rho_{\alpha_1}^{n_1} \circ \cdots \circ \rho_{\alpha_k}^{n_k}$, with $\rho_\emptyset$ the identity map. I record a few simple but useful facts about degree and order.

Lemma 1.1. 1. *The successor of θ in the order on Θ is $x_0\theta$.*

2. *Let $0 \neq r$, $s \in R^\alpha$. Then*[1]

   $\deg(rs) = \deg(s) + \deg(r)$.

3. *Let θ_0, θ_1 be standard monomials. Then*

   $\mathrm{ord}(\theta_0\theta_1) = \max\{\mathrm{ord}(\theta_0), \mathrm{ord}(\theta_1)\}$.

4. *Let θ_0, θ be standard monomials. If* $\mathrm{ord}(\theta_0) < \mathrm{ord}(\theta)$ *then* $\theta\theta_0 = \rho_\theta(\theta_0)\theta$.

Proof. The first three parts are obvious. The last part is proved by induction on the order of θ. If $\theta \in K$ there is nothing to prove. Otherwise $\theta = \theta' x_\beta^n$ with $\mathrm{ord}(\theta_0) < \beta$. Then $\theta\theta_0 = \theta' x_\beta^n \theta_0 = \theta' \rho_\beta^n(\theta_0) x_\beta^n$ (since $\mathrm{ord}(\theta_0) < \beta$, θ_0 is in the domain of ρ_β), and then since $\rho_\beta^n(\theta_0) \in K$, by induction hypothesis the latter equals $\rho_{\theta'}(\rho_\beta^n(\theta_0))\theta' x_\beta^n$, that is, it equals $\rho_\theta(\theta_0)\theta$. ■

Let $S = \{f \in R^\alpha : f$ has non zero constant term$\}$. We see from[3, Theorem 4.5] that S is easily seen to be a left Ore set in R^α and every

[1] Here '+' represents ordinal addition; the order of r and s on both sides is significant, as neither operation is commutative.

non-zero element r of R^α can be written in the form $r = s\theta$, where $s \in S$ and $\theta \in \Theta$. In fact it follows immediately from the proof of [3, Theorem 4.5] that θ is just the indeterminate part of the term of r of least degree. Let $\mathcal{R}^\alpha = (R^\alpha)_S$. Every non-zero element of $\mathcal{R}^\alpha$ can be written as a sum of terms of the form $s^{-1}\theta$ where $s \in S$ and $\theta \in \Theta$.

$\mathcal{R}^\alpha$ is a principal left ideal domain, every left ideal is generated by a monic standard monomial, every left ideal is two sided, and the left ideals are well-ordered by reverse inclusion. Thus $\mathcal{R}^\alpha$ is left FBN and so the indecomposable injective left $\mathcal{R}^\alpha$-modules are in one-to-one correspondence with the prime ideals. A typical non zero left ideal is then $\mathcal{R}^\alpha\theta$, with $\theta \in \Theta$. Then $\mathcal{R}^\alpha\theta_0 \supset \mathcal{R}^\alpha\theta_1$ if and only if $\theta_0 \prec \theta_1$.

Considerably more can be read out of the analysis of the descending sequence of left ideals in the proof of [3, Theorem 4.6] than is stated in the conclusion of the theorem. In fact we have that the Jacobson radical $J(\mathcal{R}^\alpha)$ is the ideal $J = \mathcal{R}^\alpha x_0$, and every left ideal is J^β for some β, so J^β is the β-th ideal in the descending order just described. In particular, the prime ideals of $\mathcal{R}^\alpha$ are $J, J^\omega = \mathcal{R}^\alpha x_1, \cdots, J^{\omega^n} = \mathcal{R}^\alpha x_n, \cdots, J^{\omega^\omega} = \mathcal{R}^\alpha x_\omega, \cdots, J^{\omega^\beta} = \mathcal{R}^\alpha x_\beta, \cdots, J^{\omega^\alpha} = 0$. (Once again the same model-theoretic explanation as in Sec. 1.1 could be used to show that these are exactly the prime ideals.)

For the purpose of developing explicit descriptions of the indecomposable injectives, it will be useful to have descriptions of $\mathcal{R}^\alpha/\mathcal{R}^\alpha x_\beta$ for each $\beta < \alpha$. Let $S_\beta = S \cap \bar{R}_\beta$. Then S_β is a left Ore set in $\bar{R}_\beta$. Clearly for any $s \in S$ there is unique $\bar{s} \in S_\beta$ such that $s - \bar{s} \in R^\alpha x_\beta$; similarly for any $r \in R^\alpha$ there is a unique $\bar{r} \in \bar{R}_\beta$ such that $r - \bar{r} \in R^\alpha x_\beta$. Thus for any $s^{-1}r \in \mathcal{R}^\alpha$, $s^{-1}r \equiv \bar{s}^{-1}\bar{r}\ (\mathcal{R}^\alpha x_\beta)$. So $\mathcal{R}^\alpha/\mathcal{R}^\alpha x_\beta \cong (\bar{R}_\beta)_{S_\beta}$. Here the action of $\mathcal{R}^\alpha$ on the $\mathcal{R}^\alpha$-module $(\bar{R}_\beta)_{S_\beta}$ can be thought of as "ordinary multiplication in $\mathcal{R}^\alpha$, followed by setting to 0 any indeterminate x_γ, $\gamma \geq \beta$". Furthermore, since by construction $\bar{R}_\beta$ is a left Ore domain ([3, Theorem 2.8]), we may consider $(\bar{R}_\beta)_{S_\beta}$ to be embedded in the left quotient field K^β of $\bar{R}_\beta$.

2 THE INDECOMPOSABLE INJECTIVES OVER *R*

I give explicit descriptions of the two indecomposable injective right modules over the ring R of Jategaonkar. I start with $\mathrm{E}(R/J)$. Remember that part of the point is to see that the structure of $\mathrm{E}(R/J)$ can be deduced in a fairly straightforward way from what we know about the ring. I first note that $R/J \cong F$. In fact, $\langle b, d\rangle \equiv_J \langle 0, \rho(d)\rangle$, and the action of R on R/J is just the action of the subring F of R on itself: for $a \in F$, $a = \langle 0, a\rangle/J$, so

$$a.\langle b, d\rangle = \langle 0, a\rangle\langle b, d\rangle/J = \langle \sigma(a)b, ad\rangle/J = \langle 0, ad(0)\rangle/J,$$

that is, $a.\langle b, d\rangle = a\rho(d)$.

To construct the injective envelope, note the following: By Baer's Criterion we only need to be able to solve systems of equations in one variable over R/J, and since R is a principal right ideal ring we only need to be able to solve single equations. That is, the injective hull is the divisible hull. Since $\mathrm{E}(R/J)$ is an essential extension of R/J, every element of $\mathrm{E}(R/J)$ is obtained from some element of R/J by (not necessarily unique) division. In fact, from the ideal structure of R we see that every non-zero element e of $\mathrm{E}(R/J)$ satisfies some non-zero equation of the form $e.x^n = r/J$ or $e.\langle 1, 0\rangle = r/J$.

Since $F \hookrightarrow R$, every right R-module is a right vector space over F, so we can start by trying to specify an F-basis for $\mathrm{E}(R/J)$, and by the previous comments, we can take solutions to the equations $v.x^n = 1$ ($n \geq 0$) and $v.\langle 1, 0\rangle = 1$ as a first attempt. Call these solutions $X^0 = 1,\ X^{-1},\ X^{-2}, \cdots;$ $X^{-\infty}$ respectively.

We can do better than this. D is a subring of R, $F[x] \leq D$. Also, $\big(R/J(R)\big)_D \cong (D/xD)_D$.

Lemma 2.1. *Any injective right R-module E is also an injective right D-module and an injective right $F[x]$-module.*

Proof. D and $F[x]$ are principal ideal domains, so all that we have to check is divisibility, and any equation $v.\langle 0, d\rangle = e$ over E, $d \in D$ ($d \in F[x]$) is D-consistent ($F[x]$-consistent). But $\langle 0, d\rangle$ is not a left zero-divisor in R, so this equation is also R-consistent. Hence it has a solution in E. ■

Hence $E_D(D/xD)$ is a direct summand (as a D-module) of $E_R(R/J)$. We know by a construction of Northcott[12] that $E_{F[x]}(D/xD)$ can be given an explicit description as a module of "inverse polynomials": $\sum_{i=0}^{\infty} X^{-i}F$ (as a right F-vector space), with $F[x]$-scalar multiplication determined from

$$X^{-i}.ax^j = \begin{cases} X^{-i+j}a & \text{if } -i+j \leq 0 \\ 0 & \text{otherwise} \end{cases}$$

By the results of Matlis[10] the D ($= F[x]_{\langle\langle x\rangle\rangle}$) structure on this module is given as follows: For any $d \in D$, express d as a formal power series $\sum_{i=0}^{\infty} a_i x^i$ about 0 (possible since $d(0)$ is always defined), then given $f(X^{-1}) \in F[X^{-1}]$, multiply $f(X^{-1})$ by $\sum_{i=0}^{\infty} a_i x^i$ according to the Northcott rule. This is well-defined, as only $\deg(f) + 1$ terms of the expansion of d can enter into the computation in a non-trivial way. Alternatively, as computing the power series expansion may be impractical, we can give a more computational description of the operation (which will in fact be the only possible approach in the case of the ring $\mathcal{R}^\alpha$). The element $d \in D$ can be written in the form $\frac{p(x)}{q(x)}$, a ratio of polynomials in $F[x]$ in lowest form, and $q(0) \neq 0$. Then $f(X^{-1}).d = v$ if and only if $f(X^{-1}).p(x) = v.q(x)$. It is easy to see that the solution v to this linear equation must be of degree less than or equal to $\deg(f(X^{-1}).p(x))$, and that since $q(0) \neq 0$ there must be a unique solution, which can be found by writing v in general form, multiplying out both sides of the equation, and comparing coefficients.

How does R act on its D-submodule $F[X^{-1}]$? This is easy to answer: $f(X^{-1}).\langle b, d\rangle = f(X^{-1}).\langle b, 0\rangle + f(X^{-1}).\langle 0, d\rangle$. The second term is just the D-module action already described. Now $f(X^{-1}).J^n = 0$ for some $n \in \omega$, and so since $\langle b, 0\rangle \in J^\omega$, $f(X^{-1}).\langle b, 0\rangle = 0$. That is, $f(X^{-1}).\langle b, d\rangle = f(X^{-1}).d$.

Note now that for any $a \in F$, $v.\langle 1, 0\rangle = a$ is consistent. For $\langle 1, 0\rangle\langle b, d\rangle = 0$ iff $\langle b, d\rangle \in J$ iff $a.\langle b, d\rangle = 0$. I note in passing that e solves $v.\langle b, 0\rangle = a$ iff e solves $v.\langle 1, 0\rangle = ab^{-1}$. What is the structure of the family of all solutions to equations of the the form $v.\langle 1, 0\rangle = a$?

Suppose that e is a solution to $v.\langle 1, 0\rangle = a$. Note two products:

$$e.\langle b, 0\rangle = e.\langle 1, 0\rangle\langle 0, b\rangle = a\langle 0, b\rangle = ab$$

$$e.\langle 0, d\rangle\langle 1, 0\rangle = e.\langle \sigma(d), 0\rangle = e.\langle 1, 0\rangle\langle 0, \sigma(d)\rangle = a\langle 0, \sigma(d)\rangle = a\sigma(d)$$

So, the F-subspace generated by e consists of solutions to the equations $v.\langle 1, 0\rangle = a\sigma(f)$, $f \in F$. But σ makes F into an infinite dimensional vector space over itself.

For the moment, let C denote F regarded as a right vector space over itself via the action $c.a = c\sigma(a)$.

Lemma 2.2. *If $(c_i)_{i\in I}$ is F-linearly independent in C, and for each $i \in I$, e_i is a solution in $\mathrm{E}(R/J)$ to $v.\langle 1, 0\rangle = c_i$, then $(e_i)_{i\in I}$ is F-linearly independent in E.*

Proof. Suppose that for some finite $I' \subset I$, $\sum_{i\in I'} e_i.\langle 0, a_i\rangle = 0$ where $a_i \in F$. Then $e_i.\langle 0, a_i\rangle\langle 1, 0\rangle = e_i.\langle 1, 0\rangle\langle 0, \sigma(a_i)\rangle = c_i.\langle 0, \sigma(a_i)\rangle = c_i\sigma(a_i)$. So certainly (in F), $\sum_{i\in I'} c_i\sigma(a_i) = 0$; thus in C_F, $\sum_{i\in I'} c_i.a_i = 0$. Thus by assumption, $a_i = 0$ for all $i \in I'$. ■

Lemma 2.3. *Let $(c_i)_{i\in I}$ be an F-basis for C_F, and for each $i \in I$, let e_i be a solution in $\mathrm{E}(R/J)$ of $v.\langle 1, 0\rangle = c_i$. If $a \in F$, and $e \in E$ is a solution of $v.\langle 1, 0\rangle = a$, then e is an F-linear combination of the e_i's modulo $\{f \in E : f.J^n = 0$ for some $n\}$.*

Proof. Since $a \in C_F$, we have $a = \sum_{i \in I'} c_i . a_i$ for some finite $I' \subset I$ and $a_i \in F$, $(i \in I')$. Thus, in F, $a = \sum_{i \in I'} c_i \sigma(a_i)$. Consider $\hat{e} = \sum_{i \in I'} e_i . \langle 0, a_i \rangle$. Now $\hat{e} . \langle 1, 0 \rangle = \sum_{i \in I'} e_i . \langle 0, a_i \rangle \langle 1, 0 \rangle = \sum_{i \in I'} e_i . \langle 1, 0 \rangle \langle 0, \sigma(a_i) \rangle = \sum_{i \in I'} c_i . \langle 0, \sigma(a_i) \rangle = \sum_{i \in I'} c_i \sigma(a_i) = a$. Thus e and $\hat{e}$ are both solutions to $v . \langle 1, 0 \rangle = a$, and so $(e - \hat{e}) . \langle 1, 0 \rangle = 0$, thus $(e - \hat{e}) . J^n = 0$ for some n. ■

Thus we see that the family of all solutions of equations of the form $v . \langle b, 0 \rangle = a$, $a \in F$, produces a copy (modulo the things annihilated by powers of x) of C_F. Our initial "guess" for the structure of $\mathrm{E}(R/J)$ as an F-vector space included a single basis element $X^{-\infty}$ to represent these solutions. We see now that this was wrong, but it still suggests a natural representation as a place holder for a copy of the infinite dimensional F-space C_F. We take as a canonical representation of a solution of $v . \langle 1, 0 \rangle = a$ the element $X^{-\infty} a$. The scalar multiplication is determined by the two products computed earlier: for elements $X^{-\infty} a$, $a \in F$,

$$X^{-\infty} a . \langle b, d \rangle = X^0 ab + X^{-\infty} a \sigma(d).$$

Theorem 2.4. $\mathrm{E}(R/J) \cong F[X^{-1}] \oplus X^{-\infty} F$ *with* R*-action given by*

$$(f(X^{-1}) + X^{-\infty} a) . \langle b, d \rangle = f(X^{-1}) . d + X^0 ab + X^{-\infty} a \sigma(d)$$

(*with the scalar action of* D *on* $F[X^{-1}]$ *described earlier*).

Proof. We have an independent check that the construction of the module E of the theorem is correct by actually verifying the definition: (1) this is a well-defined R-module action; (2) E is an essential extension of its R-submodule $X^0 F \cong R/J$; (3) E is a divisible right R-module. The fact that these checks are entirely routine and mechanical supports the assertion that the description given here is explicit. Of course, since part of the purpose of this section is precisely to convince the reader that the computations are, indeed, elementary, I present some of them.

1. The only thing that involves the slightest difficulty is checking the scalar associative law. We calculate:

$$\begin{aligned}
&\big((f(X^{-1}) + X^{-\infty} a) . \langle b, d \rangle\big) . \langle b', d' \rangle \\
&\quad = \big(f(X^{-1}) . d + X^0 ab + X^{-\infty} a \sigma(d)\big) . \langle b', d' \rangle \\
&\quad = \big(f(X^{-1}) . d + X^0 ab\big) . d' + X^0 a \sigma(d) b' + X^{-\infty} a \sigma(d) \sigma(d') \\
&\quad = f(X^{-1}) . (dd') + X^0 \big(ab . d' + a \sigma(d) b'\big) + X^{-\infty} a \sigma(dd')
\end{aligned}$$

We observe that the term $ab.d'$ represents the usual action of D on the right on D/xD; that is, $ab.d' = ab\rho(d')$. We now calculate:

$$\begin{aligned}&\big(f(X^{-1}) + X^{-\infty}a\big).(\langle b, d\rangle\langle b', d'\rangle)\\ &\quad= \big(f(X^{-1}) + X^{-\infty}a\big).\langle \sigma(d)b' + b\rho(d'), dd'\rangle\\ &\quad= f(X^{-1}).(dd') + X^0 a\big(\sigma(d)b' + b\rho(d')\big) + X^{-\infty}a\sigma(dd')\\ &\quad= f(X^{-1}).(dd') + X^0\big(a\sigma(d)b' + ab\rho(d')\big) + X^{-\infty}a\sigma(dd')\end{aligned}$$

2. If $f(X^{-1}) \in F[X^{-1}]$ and $\deg(f) = n$, then $f(X^{-1}).\langle 0, x^n\rangle$ is a non-zero element of X^0F. If $a \neq 0$ then $(f(X^{-1}) + X^{-\infty}a).\langle 1, 0\rangle = X^0a$, a non-zero element of X^0F.
3. We need to be able to solve $v.\langle b, d\rangle = f(X^{-1}) + X^{-\infty}a$ whenever this is consistent. If $d \neq 0$, then $\langle b, d\rangle$ is not a left zero-divisor and so $v.\langle b, d\rangle = e$ is consistent for any $e \in E$. It is easy to verify that if $g(X^{-1})$ is any solution in $F[X^{-1}]_D$ of the equation $v.d = f(X^{-1}) - X^0ab/\sigma(d)$, then $g(X^{-1}) + X^{-\infty}(a/\sigma(d))$ solves the given equation. If $d = 0$ then $\langle b, d\rangle$ is a left zero-divisor with right annihilator J; therefore $v.\langle b, 0\rangle = f(X^{-1}) + X^{-\infty}a$ is consistent iff $a = 0$ and $f(X^{-1}) = X^0c \in F$; as noted the equation $v.\langle b, 0\rangle = c$ has a solution $X^{-\infty}(c/b)$. ■

It should be noted in step (3) we could have just as easily found *all* the solutions of the given equation; this is of course another test to recognize an "explicit" description of the module in question.

The structure of the other indecomposable injective over R is easy to discern. I note that $R/J^\omega \cong D_D$, so by lemma 2.1, $\mathrm{E}_R(R/J^\omega) \supseteq \mathrm{E}_D(D) = F(X)_D$, the quotient field of D. The R-action on D is as follows: $\langle b', d'\rangle \equiv_{J^\omega} \langle 0, d'\rangle$, and $\big(\langle 0, d'\rangle/J^\omega\big).\langle b, d\rangle = \langle\sigma(d')b, d'd\rangle/J^\omega = \langle 0, d'd\rangle/J^\omega$. The D action on $F(X)$ is just ordinary multiplication in $F(X)$, and $F(X)$ becomes a right R-module by the action $q.\langle b, d\rangle = qd$.

An equation $v.\langle b, d\rangle = q$ with $q \in F(X)$ is consistent iff $d \neq 0$. For if $d \neq 0$, then $\langle b, d\rangle$ is not a left zero-divisor in R, and so the equation is consistent. On the other hand, if $d = 0$ then the right annihilator of $\langle b, d\rangle$ in R is J; $q.J = 0$ iff $q = 0$. Thus the only consistent equations are $v.\langle b, d\rangle = q$, $d \neq 0$, and by the multiplication described these have solutions in $F(X)$, namely $v = q/d$.

Theorem 2.5. $\mathrm{E}(R/J^\omega) \cong F(x)$ *under the scalar action* $q.\langle b, d\rangle = qd$. ■

Note that $\mathrm{E}(R/J)/F[X^{-1}] \cong_R \mathrm{E}(R/J^\omega)$.

3 SOME APPLICATIONS OF THE EXPLICIT DESCRIPTION

The true test of the success of an "explicit description" is that one should be able to use that description—in a straightforward way—to learn various interesting things about the module in question. I explore some of these things in this section for the two modules constructed in the previous section. It is clear that solving systems of equations should be a purely mechanical task, and I leave this for the reader to verify for her/himself.

Exercise 3.1. Verify that the following system of linear equations over $\mathrm{E}(R/J)$ is inconsistent.

$$u.\langle 1, 1\rangle + v.\langle 0, x\rangle = X^{-2} + X^{-\infty}$$
$$u.\langle 0, x\rangle + v.\langle 1, x^2\rangle = X^{-1}x_1 + X^{-\infty}$$

Exercise 3.2. Find all solutions to the following system of linear equations over $\mathrm{E}(R/J)$.

$$u.\langle 0, x\rangle + v.\langle 1, 0\rangle = X^{-\infty}$$
$$u + v.\langle 1, 1\rangle = X^0 - X^{-\infty}$$

3.1 Series in $E(R/J)$

We try to understand the structure of a complicated module by viewing it as the union of a continuous ascending chain of submodules, such that the quotients of successive pairs of modules in the chain are uncomplicated in some way. I will look at several different series here.

The *socle* of a module is defined to be the sum of its simple submodules. This can be extended to a series (*socle series*, *Loewy series*) as follows: for any ordinal α, given $\mathrm{soc}_\alpha(M)$, $\mathrm{soc}_{\alpha+1}(M)$ is the full inverse image in M of $\mathrm{soc}(M/\mathrm{soc}_\alpha(M))$. It is easy to see that, for $n < \omega$, $\mathrm{soc}_n(\mathrm{E}(R/J)) = \sum_{i<n} X^{-i}F$, and so $\mathrm{soc}_\omega(\mathrm{E}(R/J)) = F[X^{-1}]$. However, there are no minimal submodules over $F[X^{-1}]$. For instance, $\langle\langle X^{-\infty}x_1\rangle\rangle \supset \langle\langle X^{-\infty}{x_1}^2\rangle\rangle \supset \langle\langle X^{-\infty}{x_1}^3\rangle\rangle \supset \cdots$ is an infinite proper descending chain of submodules not contained in $F[X^{-1}]$ but whose intersection is (and all other submodules which contain $F[X^{-1}]$ are part of a similar descending sequence). Thus the socle series stabilizes at soc_ω, and the socle series fails to capture all the structure of $\mathrm{E}(R/J)$.

The *fundamental series* (see, e.g.,[4, Chapter 9.1]) is defined only for two-sided noetherian rings but we can try to follow through the definition anyway and see what happens. The main failure is the failure of the *incomparability condition*, and we can see the disastrous effect that this has on any attempt to analyze the structure of $E = \mathrm{E}(R/J)$ in this way. We see that (in the notation of[4]) $\mathrm{F}_0(E) = \{0\}$, $\Omega_1(E) = \mathrm{Ass}(E) = \{J\}$, $\mathrm{F}_1(E) = \{e \in E\colon r(eR) \supseteq J\} = X^0F$, $\Omega_2(E) = \mathrm{Ass}(E/F_1) = \{J, J^\omega\}$, $\mathrm{F}_2(E) = \{e \in E\colon r\big((eR + F_1)/F_1\big) \supseteq J^\omega\} = E$. Because $J^\omega \subset J$, all the J-structure is lost after the first step. Note that J^ω is a right second layer link of J^ω to J, the so called "undesirable case" of Jategaonkar's "Main Lemma", since $J^\omega \subset J$.

The *elementary socle series* is a first-order definable analogue of the socle series. It was introduced by Herzog in[2, 10.2]. It is defined by setting $\mathrm{soc}^0(M) = 0$, and letting $\mathrm{soc}^{\alpha+1}(M)$ be the sum of $\mathrm{soc}^\alpha(M)$ and all the first order definable subgroups N of M which are minimal with respect to the property $N \not\subseteq \mathrm{soc}^\alpha(M)$, and making the series continuous at limit ordinals. Each $\mathrm{soc}^\alpha(M)$ is a submodule, in fact a definably closed submodule, of M. If M is a *totally transcendental* module (equivalently a Σ-pure-injective module), the elementary socle series is well-defined in M and exhausts M([2,7]). Every injective right module over a (right) noetherian ring is totally transcendental, and the definable subgroups are just the solution sets of finite homogeneous systems of linear equations. So it is easy to see that

$$\begin{aligned} \mathrm{soc}^0(E) &= \{0\} \\ \mathrm{soc}^n(E) &= \sum_{i<n} X^{-i}F \\ \mathrm{soc}^\omega(E) &= F[X^{-1}] \\ \mathrm{soc}^{\omega+1}(E) &= E \end{aligned}$$

While there are no minimal proper submodules over $F[X^{-1}]$, the only definable subgroup over $F[X^{-1}]$ is E itself. It is interesting to note that in this case, at least, the elementary socle series seems to be quite well behaved: $\mathrm{soc}^{n+1}(E)/\mathrm{soc}^n(E) \cong F$, and the structure inherited from E is just that of a right F-vector space. $\mathrm{soc}^{\omega+1}(E)/\mathrm{soc}^\omega(E) \cong F$ as well, but this time F inherits the right σ-structure, i.e., what we apparently have here is the D-module B_D. But more structure than that is definable on B_D: notice that the equation $v\,\boldsymbol{.}\,\langle 0, d\rangle = X^{-\infty}a$ (for $d \neq 0$) has the unique solution $v = X^{-\infty}\big(a/\sigma(d)\big)$ modulo $\mathrm{soc}^\omega(E)$. Thus in fact σ extends (in a definable way) to an action making F a right $F(x)$-vector space. This is just another way of describing the fact already mentioned that $\mathrm{E}(R/J)/F[X^{-1}] \cong_R \mathrm{E}(R/J^\omega)$.

3.2 Endomorphism Rings

Given the descriptions of $\mathrm{E}(R/J)$ and $\mathrm{E}(R/J^{\omega})$, it is routine to find descriptions of their endomorphism rings.

Proposition 3.3. $\mathrm{End}(\mathrm{E}(R/J^{\omega})) = F(x)$. ■

The endomorphism ring of $\mathrm{E}(R/J)$ is another matter entirely. As a D-module, $\mathrm{E}(R/J)$ is the injective module $\mathrm{E}(D/xD) \oplus \mathrm{E}(D)$. Then $\mathrm{End}(\mathrm{E}(R/J)_D)$ can be thought of as a generalized 2×2 matrix ring. We analyze the components of this matrix ring as follows: We have

$$\mathrm{End}(\mathrm{E}(D/xD)) \cong \mathrm{End}\big(F[X^{-1}]_D\big) \cong F[[x]]$$

where $F[[x]]$ has the natural (Northcott) action on $F[X^{-1}]$. Also $\mathrm{End}(\mathrm{E}(D)) \cong \mathrm{End}(F(x)_D) \cong F(x)$. Clearly $\mathrm{Hom}_D(\mathrm{E}(D), \mathrm{E}(D/xD)) = 0$. The only tricky piece to describe in a useful way is

$$M = \mathrm{Hom}_D(\mathrm{E}(D), \mathrm{E}(D/xD)) = \mathrm{Hom}_D(F(x), F[X^{-1}])\,,$$

which is quite big. One convenient way of representing M is as the group $F[x^{-1}, x]]$ of *formal Laurent series* over F. We have to show how to make this into an $(F[[x]], F(x))$-bimodule, and explain how the elements of M act as homomorphisms. Given a formal Laurent series $\zeta = \sum_{j \geq n_0} x^j b_j$ we define $\varphi : F(x) \longrightarrow F[X^{-1}]$ as follows: For k such that $k \geq 0$ and $k \geq n_0$, let $e_k = \sum_{n_0 \leq j \leq k} X^{j-k} b_j \in F[X^{-1}]$. For any $u \in F(x)$ there is $n \geq 0$ such that $x^n u \in D$. Define $\varphi(u) = e_n \mathbin{.} (x^n u)$, where "." is the usual scalar action of D on $F[X^{-1}]$. I leave it to the reader to check that the definition of φ does not depend on the choice of n (sufficiently large) and that φ is a homomorphism. Furthermore, if ψ is any homomorphism, we can see that ψ is induced from the Laurent series ξ defined as follows: $\xi = \sum_{j \geq n_0} x^j b_j$ where $\psi(1) = \sum_{n_0 \leq j \leq 0} X^j b_j$, and for $k > 0$, b_k is the coefficient of X^0 in $\psi(\frac{1}{x^k})$. The left $F[[x]]$-action on M is just ordinary multiplication of power series. The right $F(x)$-action is also straightforward: if $q \in F(x)$ then $x^n q \in D$ for some $n \in \omega$. Now any element of D can be written as a formal power series, $x^n q = \sum_{i \geq 0} a_i x^i$. Thus for any formal Laurent series $\zeta = \sum_{j \geq n_0} x^j b_j$,

$$\zeta \mathbin{.} q = \left(\zeta \frac{1}{x^n}\right) \mathbin{.} (x^n q) = \left(\sum_{j \geq n_0} x^{j-n} b_j\right)\left(\sum_{i \geq 0} x^i a_i\right).$$

So we obtain the following:

Lemma 3.4.

$$\mathrm{End}_D(\mathrm{E}(R/J)) \cong \begin{pmatrix} F[[x]] & {}_{F[[x]]}F[x^{-1}, x]]_{F(x)} \\ 0 & F(x) \end{pmatrix}$$

with the actions on $F[x^{-1}, x]]$ *as described.* ■

Quite a few computations have been left out of the above discussion, and the reader is warned that a detailed checking of these claims is, while routine, quite tedious.

Now we have to identify the R-endomorphisms among the D-endomorphisms. We represent an element of $\mathrm{E}(R/J)$ as a column vector with entries $f(X^{-1})$ and a, letting the endomorphism ring above act on these from the left by ordinary matrix multiplication. Again, equally routine and tedious computations yield:

Lemma 3.5. *The elements of* $\mathrm{End}_D(\mathrm{E}(R/J))$ *which are R-endomorphisms are exactly those of the form*

$$\begin{pmatrix} \xi & \zeta \\ 0 & \xi(0) \end{pmatrix}.$$

So we see that $\mathrm{End}_R(\mathrm{E}(R/J))$ is again a trivial extension.

Theorem 3.6. *Let* $S = F[[x]] \oplus F[x^{-1}, x]]$ *with* $F[x^{-1}, x]]$ *the* $(F[[x]], F(x))$-*bimodule described earlier. Then S is a ring with multiplication*

$$\langle \xi, \zeta \rangle \langle \xi', \zeta' \rangle = \langle \xi\xi', \xi . \zeta' + \zeta . \xi'(0) \rangle .$$

S is isomorphic to $\mathrm{End}_R(\mathrm{E}(R/J))$, *acting on* $\mathrm{E}(R/J)$ *from the left according to the rule*

$$\langle \xi, \zeta \rangle . (f(X^{-1}) + X^{-\infty}a) = \xi . f(X^{-1}) + X^0 \zeta . a + X^{-\infty}\xi(0)a. \qquad ■$$

3.3 Elementary Duals

Prest[13, 8.4] (see also[14] and Herzog[2]), introduced a concept of *elementary duality* between right and left modules which lives on the lattice of positive primitive formulas of the languages of right and left R-modules. In some sufficiently well-behaved cases, this extends to a duality between the left and right indecomposable pure-injective modules. This duality makes indecomposable injective right modules correspond to indecomposable pure-injective flat left modules.[2, Cor. 9.6] In this case the elementary duals of the indecomposable injective right R-modules are easy to construct. One simply considers the rather uncomplicated set of elementary invariants (pp-pairs) in each injective. Once again, although deriving the structure of the duals is quite lengthy, once a description has been produced, it is straightforward to check that the answer is correct by performing the proper computations.

For all matters relating to the model theory of modules, I refer the reader to[13].

In the following, let η be the natural map of $D = F[x]_{(x)}$ into $F[[x]]$, that is, $\eta(d)$ is the representation of $d \in D$ as a formal power series. Extending the earlier use of the symbol, ρ also represents evaluation at 0 for elements of $F[[x]]$, that is, $\rho(f)$ is the constant term of $f \in F[[x]]$.

Theorem 3.7. 1. *The elementary dual of the right R-module* $\mathrm{E}(R/J)$ *is the flat indecomposable pure-injective left R-module $_RP = F \oplus F[[x]]$, with the action of R given by $\langle b, d\rangle.\langle a, f\rangle = \langle b\rho(f) + \sigma(d)q, \eta(d)f\rangle$.*

2. *The elementary dual of the right R-module* $\mathrm{E}(R/J^{\omega})$ *is the flat indecomposable pure-injective left R-module $F(X)$, with the action of R given by $\langle b, d\rangle q = dq$.*

Proof. I first have to verify that $_RP$ is indeed a left R-module under the scalar multiplication described. As usual, the only minor difficulty is in checking the scalar associative law:

$$\begin{aligned}
&(\langle b, d\rangle\langle b', d'\rangle)\,.\,\langle q, f\rangle\\
&\quad= \langle b\rho(d') + b'\sigma(d),\, dd'\rangle\,.\,\langle q, f\rangle\\
&\quad= \Big\langle (b\rho(d') + b'\sigma(d))\rho(f) + \sigma(dd')q,\, \eta(dd')f\Big\rangle\\
&\quad= \big\langle b\rho(d')\rho(f) + b'\sigma(d)\rho(f) + \sigma(d)\sigma(d')q,\, \eta(d)\eta(d')f\big\rangle
\end{aligned}$$

$$\begin{aligned}
&\langle b, d\rangle\,.\,(\langle b', d'\rangle\,.\,\langle b', d'\rangle)\\
&\quad= \langle b, d\rangle\,.\,\langle b'\rho(f) + \sigma(d')q,\, \eta(d')f\rangle\\
&\quad= \Big\langle b\rho(\eta(d')f) + \sigma(d)(b'\rho(f) + \sigma(d')q),\, \eta(d)(\eta(d')f)\Big\rangle\\
&\quad= \langle b\rho(d')\rho(f) + \sigma(d)b'\rho(f) + \sigma(d)\sigma(d')q,\, \eta(d)\eta(d')f\rangle
\end{aligned}$$

Observe that $\rho(\eta(d')) = \rho(d')$.

Next we verify that P is flat. There are natural characterizations of flatness in terms of solution sets to systems of linear equations [13, 14.6, 14.8], but at least in this case a characterization in terms of tensors allows us to use the information that we have available most efficiently. Now ${}_RP$ is flat if and only if the natural homomorphism $I \otimes_R P \longrightarrow P$ is a monomorphism for every finitely generated right ideal I [15, Prop. 10.6]. Since the right ideals of R are all principal, all that needs to be checked is that

$$\langle 0,\, x^n\rangle . \langle q, f\rangle = 0 \Longrightarrow \langle 0,\, x^n\rangle \otimes \langle q, f\rangle = 0$$

and that

$$\langle 1,\, 0\rangle . \langle q, f\rangle = 0 \Longrightarrow \langle 1,\, 0\rangle \otimes \langle q, f\rangle = 0\,.$$

If $0 = \langle 0,\, x^n\rangle . \langle q, f\rangle = \langle {x_1}^n q,\, x^n f\rangle$, then clearly $\langle q, f\rangle = 0$. If $0 = \langle 1,\, 0\rangle . \langle q, f\rangle = \langle \rho(f),\, 0\rangle$, then $f = xf'$ and $\langle q, f\rangle = \langle 0,\, x\rangle\langle \frac{q}{x_1}, f'\rangle$. Thus $\langle 1,\, 0\rangle \otimes \langle q, f\rangle = \langle 1,\, 0\rangle \otimes \langle 0,\, x\rangle . \langle \frac{q}{x_1}, f'\rangle = \langle 1,\, 0\rangle\langle 0,\, x\rangle \otimes \langle \frac{q}{x_1}, f'\rangle = 0 \otimes \langle \frac{q}{x_1}, f'\rangle = 0$.

Next I verify that P is pure-injective. Since ${}_RP$ is flat and R is right noetherian, it follows by[13, 14.17] that every pp-definable subgroup of P has the form LP for some finitely generated right ideal L of R. Thus the pp-definable subgroups of ${}_RP$ are exactly $0 \subset \langle 1,\, 0\rangle P \subset \cdots \subset \langle 0,\, x^n\rangle P \subset \cdots \subset \langle 0,\, x^2\rangle P \subset \langle 0,\, x\rangle P \subset P$. Thus the only infinite consistent descending chains of congruences are essentially those of the form $\{\langle 0,\, x^n\rangle \mid v - \langle q_i,\, \sum_{i<n} a_i x^i\rangle : n < \omega\}$, for some sequence $(a_i)_{i<\omega}$ in F; and this has solutions $\langle q,\, \sum_{i<\omega} a_i x^i\rangle$ for any $q \in F$. Thus ${}_RP$ is pure-injective.

Now I have to verify that ${}_RP$ is indecomposable. For this it suffices to show that it is pure-uniform, that is, that any two non-zero elements are linked by a non-trivial pp relation. That is, if $0 \neq p, p' \in P$, there is a pp-formula $\varphi(u, v)$ such that $P \models \varphi[p, p'] \wedge \neg\varphi[p, 0]$. There are, unfortunately, five cases to consider. Let $p = \langle q, f\rangle$ and $p' = \langle q', f'\rangle$. For the cases where at least one of f or f' is non-zero, let ax^m and $a'x^{m'}$ be the first non-zero term of f, f' respectively. Let t' be such that q' can be written in the form $q' = r'/({x_1}^{t'} s')$, $x_1 \nmid r'$, $x_1 \nmid s'$. Then p and p' are linked by one of the following four formulas $\varphi(u, v)$:

1. $\exists u_1 v_1 w[u = \langle 0,\, ax^m\rangle . u_1 \wedge v = \langle 0,\, a'x^{m'}\rangle . v_1 \wedge \langle 0,\, x\rangle . w = u_1 - v_1]$
2. $\exists u_1 v_1 w[u = \langle 0,\, ax^m\rangle . u_1 \wedge v = \langle 0,\, a'x^{m'}\rangle . v_1 \wedge \langle 1,\, 0\rangle . w = u_1 - v_1]$
3. $\exists u_1 v_1[u = \langle 0,\, ax^m\rangle . u_1 \wedge v = \langle 0,\, a'x^{m'}\rangle . v_1 \wedge 0 = u_1 - v_1]$
4. $\exists u_1[u = \langle 0,\, ax^m\rangle . u_1 \wedge \langle {x_1}^t q',\, 0\rangle . u_1 = \langle 0,\, x^t\rangle . v]$

I leave it to the reader to sort out the various possibilities. Finally, if both f and f' are 0, chose t, r, and s for q in the same way as t', etc., were chosen for q'. Then p and p' are linked by

5. $\langle 0,\, x^{t}\sigma^{-1}(s)\sigma^{-1}(r')\rangle\,.\,u = \langle 0,\, x^{t'}\sigma^{-1}(s')\sigma^{-1}(r)\rangle\,.\,v.$

Finally, all that remains is to check the *elementary invariants* for ${}_RP$, cf.[2]. (The elementary invariants measure the index of one definable subgroup in another. The dual invariants of the dual module are equal to the invariants of the original.) The non-trivial invariants for $\mathrm{E}(R/J)$ are of the forms

$$[(v\,.\,\langle 0,\, x^{n+1}\rangle = 0)\,/\,(v\,.\,\langle 0,\, x^{n}\rangle = 0)]\,,\ (n < \omega),$$

$$[(v\,.\,\langle 1,\, 0\rangle = 0)\,/\,(v\,.\,\langle 0,\, x^{n}\rangle = 0)]\,,\ (n < \omega),$$

and

$$[(v = v)\,/\,(v\,.\,\langle 1,\, 0\rangle = 0)].$$

Each of these is easily seen to be infinite. The corresponding dual invariants are then those of the forms

$$[(\langle 0,\, x^{n}\rangle \mid v)\,/\,(\langle 0,\, x^{n+1}\rangle \mid v)]\,,\quad (n < \omega),$$

$$[(\langle 0,\, x^{n}\rangle \mid v)\,/\,(\langle 1,\, 0\rangle \mid v)]\,,\quad (n < \omega),$$

and

$$[(\langle 1,\, 0\rangle \mid v)\,/\,(v = 0)].$$

By the flatness of ${}_RP$ and the fact that all right ideals of R are principal, these are the only invariants that we need to check.

Consider any $\langle b,\, d\rangle \in R$. If $d \neq 0$ then for some $n \in \omega$, $\eta(d) = x^{n}d'$, $x \nmid d'$. Then $\langle b,\, d\rangle P = F \oplus x^{n}F[[x]]$. Furthermore, if $b \neq 0$, then $\langle b,\, 0\rangle P = F \oplus 0$. Thus all the dual invariants are seen to be infinite in ${}_RP$.

The computations required to check the second part of the theorem are trivial by comparison, and are left to the reader. ■

3.4 The Lattice of Submodules

The indecomposable injective $\mathrm{E}(R/J))$ has quite a complicated lattice of submodules, which I will not attempt to describe completely. But the explicit description is strong enough to allow us to discover a few interesting things about the lattice of submodules fairly easily. For a module M, I let $\mathcal{S}(M)$ denote the lattice of submodules of M.

Lemma 3.8. *$\mathcal{S}(\mathrm{E}(R/J^{\omega}))$ is isomorphic to the chain $1 + \mathbb{Z} + 1$.*

Proof. The algebraic structure on $\mathrm{E}(R/J^{\omega})$ is just that of $F(x)_D$, where $D = F[x]_{\langle\langle x\rangle\rangle}$. The D-submodules are just

$$0 \subset \cdots \subset x^2D \subset xD \subset D \subset (1/x)D \subset \cdots \subset \mathrm{E}(R/J^{\omega}). \qquad \blacksquare$$

Note that $F[X^{-1}] \subset \mathrm{E}(R/J)$ is uniserial with submodule lattice isomorphic to $\omega + 1$ and recall that $\mathrm{E}(R/J)/F[X^{-1}]$ is isomorphic to $\mathrm{E}(R/J^{\omega})$. So I have:

Lemma 3.9. *The following is a maximal chain in $\mathcal{S}(\mathrm{E}(R/J))$:*

$$0 \subset X^0F \subset \cdots \subset \sum_{i<n} X^{-i}F \subset \cdots \subset F[X^{-1}] \subset \cdots$$
$$\cdots \subset F[X^{-1}] + \langle\langle X^{-\infty}x_1^z\rangle\rangle \subset \cdots \subset \mathrm{E}(R/J)$$

where $n \in \omega$ and $z \in \mathbb{Z}$. $\blacksquare$

However, and in spite of the fact that $\mathrm{E}(R/J)$ is the injective envelope of a uniserial module over a uniserial ring, $\mathcal{S}(\mathrm{E}(R/J))$ is quite complicated. For instance another maximal chain is

$$0 \subset X^0F \subset \cdots \subset \langle\langle X^{-\infty}x_1^z\rangle\rangle \subset \cdots \subset X^0F + X^{-\infty}F \subset$$
$$\cdots \subset \sum_{i<n} X^{-i}F + X^{-\infty}F \subset \cdots \subset \mathrm{E}(R/J)\,.$$

The reader should then discern a sublattice isomorphic to $1 + ((\omega + 1) \times (1 + \mathbb{Z} + 1))$ lurking in the background. But although the maximal chains of this sublattice are maximal in $\mathcal{S}(\mathrm{E}(R/J))$ there is a lot more to this lattice. In fact $\mathcal{S}(\mathrm{E}(R/J))$ is not distributive: I leave it to the reader to verify that the five submodules $\langle\langle X^{-\infty}x_1\rangle\rangle$, $\langle\langle X^{-1}, X^{-\infty}x_1\rangle\rangle$, $\langle\langle X^{-1} + X^{-\infty}\rangle\rangle$, $\langle\langle X^{-\infty}\rangle\rangle$, and $\langle\langle X^{-1}, X^{-\infty}\rangle\rangle$ form an $\mathbf{M_3}$ in $\mathcal{S}(\mathrm{E}(R/J))$. There are of course similar patterns for each X^{-n} and each $X^{-\infty}x_1^z$:

Lemma 3.10. *For any $n \in \omega$ and $z \in \mathbb{Z}$, the following five submodules form a sublattice isomorphic to $\mathbf{M_3}$ in $\mathcal{S}(\mathrm{E}(R/J))$:*

$\langle\langle X^{-n}, X^{-\infty}x_1^z\rangle\rangle$, $\langle\langle X^{-(n+1)}, X^{-\infty}x_1^z\rangle\rangle$, $\langle\langle X^{-n}, X^{-(n+1)} + X^{-\infty}x_1^{z-1}\rangle\rangle$, $\langle\langle X^{-n}, X^{-\infty}x_1^{z-1}\rangle\rangle$, and $\langle\langle X^{-(n+1)}, X^{-\infty}x_1^{z-1}\rangle\rangle$. $\blacksquare$

3.5 Solutions to the Exercises

Solution 3.1. (Remember that these are right modules!) It is enough to find a right annihilator of the coefficient matrix which is not a right annihilator of the constants:

$$\begin{bmatrix} \langle 1,\, 1\rangle & \langle 0,\, x\rangle \\ \langle 0,\, x\rangle & \langle 1,\, x^2\rangle \end{bmatrix} \begin{bmatrix} \langle -x_1,\, -x^2\rangle \\ \langle 1,\, x\rangle \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} X^{-2} + X^{-\infty} & X^{-1}x_1 + X^{-\infty} \end{bmatrix} \begin{bmatrix} \langle -x_1,\, -x^2\rangle \\ \langle 1,\, x\rangle \end{bmatrix} \neq [0]$$

Solution 3.2.

$$u = X^0\left(\frac{1}{x_1} + 2 - c\right) + X^{-1}\left(\frac{1}{x_1} + 1\right) + X^{-\infty}\left(\frac{1}{x_1}\right)$$
$$v = X^0 c - X^{-1}\left(\frac{1}{x_1} + 1\right) - X^{-\infty}\left(\frac{1}{x_1} + 1\right)$$
$$(c \in F)$$

4 THE INDECOMPOSABLE INJECTIVES OVER $\mathcal{R}^\alpha$

We have already seen in Sec. 1.2 that the ring $\mathcal{R}^\alpha$ is left FBN and the prime ideals are exactly the ideals $J^{\omega^\beta} = \mathcal{R}^\alpha x_\beta$ for $\beta < \alpha$, and $J^{\omega^\alpha} = 0$, (where J is the Jacobson radical of $\mathcal{R}^\alpha$). Thus the indecomposable injectives of $\mathcal{R}^\alpha$ are precisely the injective envelopes $\mathrm{E}(\mathcal{R}^\alpha / J^{\omega^\beta})$, $\beta \leq \alpha$. By methods entirely similar to those outlined in detail in the previous section, we are led to explicit descriptions of these modules. Of course these also provide explicit descriptions of some indecomposable injective left R^α-modules, but it seems that it is unlikely that we would be able to identify in some uniform way *all* of the indecomposable injective left R^α-modules. Nonetheless, it seems reasonable that we should be able to describe explicitly any $\mathrm{E}_{R^\alpha}(R^\alpha/I)$ where we have reasonable descriptions of I and of R^α/I.

For each monic standard monomial $\theta \in \Theta$, $\theta = x_{\alpha_1}^{n_1} \cdots x_{\alpha_k}^{n_k}$, let θ^{-1} denote the formal expression $X_{\alpha_k}^{-n_k} \cdots X_{\alpha_1}^{-n_1}$ (an "inverse standard monomial"). (For $\theta = 1$ let θ^{-1} denote 1 as well.) The *order* of θ^{-1} is the order of θ. It is intended that in each indecomposable $\mathrm{E}(\mathcal{R}^\alpha/\mathcal{R}^\alpha x_\beta)$, $a\theta^{-1}$ will represent a canonical solution to a consistent equation $\theta . v = a$, where $\theta \geq x_\beta$ and $a \in K^\beta$.

The first (and major) task will be to describe $\mathrm{E}(\mathcal{R}^\alpha/J)$.

4.1 The "Biggest" Indecomposable $\mathrm{E}(\mathcal{R}^{\alpha}/J)$

Using the maps ρ_θ, $\theta \in \Theta$, K may be made into a vector space over itself since each ρ_θ restricts to a proper embedding of K into itself. Let K_θ denote the left K-vector space with underlying set K and scalar action $a \centerdot k = \rho_\theta(a)k$. Let E_0 be the K-vector space $\oplus_{\theta\in\Theta} K_\theta \theta^{-1}$. The elements of E_0 have a naturally defined degree inherited from the degree defined on R^{α}. I will show that E_0 is in fact the underlying K-vector space of $\mathrm{E}(\mathcal{R}^{\alpha}/J)$. Note that since $\mathcal{R}^{\alpha}$ is a domain, any equation $r \centerdot v = a$ with $a \in \mathcal{R}^{\alpha}/J$ and $r \neq 0$ is consistent.

Lemma 4.1. *Let θ and η be standard monomials, and let $a \in \mathcal{R}^{\alpha}/J \cong K$. Let e be any solution in $\mathrm{E}(\mathcal{R}^{\alpha}/J)$ to $\theta \centerdot v = a$. Then:*

1. $\eta \centerdot a = \eta a$ *if* $\deg(\eta) = 0$ *and* $\eta \centerdot a = 0$ *if* $\deg(\eta) > 0$.
2. *If* $\mathrm{ord}(\eta) < \mathrm{ord}(\theta)$ *then* $\eta \centerdot e$ *is a solution to* $\theta \centerdot v = \rho_\theta(\eta)a$.
3. *If* $\mathrm{ord}(\eta) = \mathrm{ord}(\theta)$, *say* $\eta = \hat{\eta}x_\gamma^m$ *and* $\theta = \hat{\theta}x_\gamma^n$, *and* $m \leq n$ *then* $\eta \centerdot e = \hat{\eta} \centerdot \hat{e}$, *where* $\hat{e}$ *is a solution to* $\hat{\theta}x_\gamma^{n-m} \centerdot v = a$.
4. *if* $\mathrm{ord}(\eta) = \mathrm{ord}(\theta)$ *as in (iii) and* $m > n$ *or if* $\mathrm{ord}(\eta) > \mathrm{ord}(\theta)$, *then* $\eta \centerdot e = 0$.

Proof.

1. $\eta \in K$ if $\deg(\eta) = 0$ and $\eta \in J$ if $\deg(\eta) > 0$.
2. Let $\eta \centerdot e = f$. Then $\theta \centerdot f = \theta \centerdot (\eta \centerdot e) = (\theta \centerdot \eta) \centerdot e = (\rho_\theta(\eta)\theta) \centerdot e$, by Lemma 1.1, which equals $\rho_\theta(\eta) \centerdot (\theta \centerdot e) = \rho_\theta(\eta) \centerdot a = \rho_\theta(\eta)a$, by part 1.
3. Let $\hat{e} = x_\gamma^m \centerdot e$. Then $\hat{\theta}x_\gamma^{n-m} \centerdot \hat{e} = \hat{\theta}x_\gamma^{n-m} \centerdot (x_\gamma^m \centerdot e) = (\hat{\theta}x_\gamma^{n-m}x_\gamma^m) \centerdot e = \theta \centerdot e = a$, and $\eta \centerdot e = \hat{\eta}x_\gamma^m \centerdot e = \hat{\eta} \centerdot (x_\gamma^m \centerdot e) = \hat{\eta} \centerdot \hat{e}$.
4. If $\mathrm{ord}(\eta) = \mathrm{ord}(\theta)$ and $m > n$ then $\eta \centerdot e = \hat{\eta}x_\gamma^m \centerdot e = \hat{\eta}x_\gamma^{m-n} \centerdot (x_\gamma^n \centerdot e)$, and $x_\gamma^n \centerdot e$ is a solution to $\hat{\theta} \centerdot v = a$ with $\mathrm{ord}(\hat{\theta}) < \mathrm{ord}(\hat{\eta}x_\gamma^{m-n})$, so we can assume without loss of generality that $\mathrm{ord}(\eta) > \mathrm{ord}(\theta)$. Thus $\eta\theta = \rho_\eta(\theta)\eta$ by Lemma 1.1. Let $f = \eta \centerdot e$. Then $\rho_\eta(\theta) \centerdot f = \rho_\eta(\theta) \centerdot (\eta \centerdot e) = (\rho_\eta(\theta)\eta) \centerdot e = (\eta\theta) \centerdot e = \eta \centerdot (\theta \centerdot e) = \eta \centerdot a = 0$, by part 1. But $\rho_\eta(\theta)$ is invertible, so $f = 0$. ■

Corollary 4.2. *If* $\mathrm{ord}(\eta) = \mathrm{ord}(\theta)$ *and* $m < n$ *then* $\eta \centerdot e$ *is a solution to* $\hat{\theta}x_\gamma^{n-m} \centerdot v = \rho_{\hat{\theta}}(\rho_\gamma^{n-m}(\hat{\eta}))a$.

Proof. By part 3 and part 2, since if $m < n$ then $\mathrm{ord}(\hat{\eta}) < \mathrm{ord}(\hat{\theta}\mathrm{x}_\gamma^{\mathrm{n-m}})$. ■

With the intent that $a\theta^{-1}$ is supposed to represent a canonical solution to the (consistent) equation $\theta \centerdot v = a$, I define an action of the standard monomials on the inverse standard monomials by recursion on the order of η.

Definition 4.3.

$$\eta\,.\,a\theta^{-1} = \begin{cases} \rho_\theta(\eta)a\theta^{-1} & \text{ord}(\eta) = -1 \text{ or } \text{ord}(\eta) < \text{ord}(\theta), \\ \hat{\eta}\,.\,aX_\gamma^{m-n}\hat{\theta}^{-1} & \text{ord}(\eta) = \text{ord}(\theta), \eta = \hat{\eta}x_\gamma^m \textit{ and } \theta = \hat{\theta}x_\gamma^n, m \leq n, \\ 0 & \text{ord}(\eta) = \text{ord}(\theta), m > n, \text{ or } \text{ord}(\eta) > \text{ord}(\theta). \end{cases}$$

Then I extend this to an action of R^α on E_0 by distributivity. It follows immediately from Lemma 4.1 that this multiplication respects the intended interpretation of the inverse monomials.

Corollary 4.4. *In the second case of the definition, if* $\text{ord}(\eta) = \text{ord}(\theta)$ *and* $m < n$ *then*

$$\eta\,.\,a\theta^{-1} = \rho_{\hat{\theta}}(\rho_\gamma^{n-m}(\hat{\eta}))aX_\gamma^{m-n}\hat{\theta}^{-1}. \qquad \blacksquare$$

Note that the condition "$\text{ord}(\eta) = -1$" must be included in the first clause of the definition as in this case there simply is no 'γ'; this is in accordance with the θ-component of E_0 being the K-vector space K_θ. The only thing that remains in order to see that E_0 is an R^α-module is to verify the associativity of the operation just defined.

Lemma 4.5. *The action of R^α on E_0 is associative, and hence makes E_0 into a left R^α-module.*

Proof. Take standard monomials $\eta_0 = \hat{\eta}_0 x_\delta^l$ and $\eta_1 = \hat{\eta}_1 x_\gamma^m$, and inverse standard monomial $a\theta^{-1} = aX_\beta^{-n}\hat{\theta}^{-1}$. We need to verify that $\eta_0\,.\,(\eta_1\,.\,a\theta^{-1}) = (\eta_0\eta_1)\,.\,a\theta^{-1}$. We have to consider a variety of cases depending on the ordering of $\{\beta, \gamma, \delta\}$, and, where appropriate, the comparisons among the exponents l, m, and n. In any case where δ or γ is greater than β, or one of them equals β and the corresponding exponent is greater than n, both sides of the associative law are eaily seen to reduce to 0. I leave it to the reader to verify that the various degenerate subcases arising from the order of η_0 or of η_1 being -1, or possibilities like $m - n = 0$ in case (2), are either trivial or handled correctly by the following computations. So we assume as induction hypothesis that

$$\eta'_0\,.\,(\eta'_1\,.\,a\theta^{-1}) = (\eta'_0\eta'_1)\,.\,a\theta^{-1}$$

for all $a\theta^{-1}$, whenever $\text{ord}(\eta'_0) \leq \text{ord}(\eta_0)$ and $\text{ord}(\eta'_1) \leq \text{ord}(\eta_1)$, with at least one of the inequalities being strict.

Case 1 ($\delta, \gamma < \beta$)

$$\begin{aligned}\eta_0 . (\eta_1 . a\theta^{-1}) &= \eta_0 . \rho_\theta(\eta_1) a\theta^{-1} && (\text{since } \gamma < \beta)\\ &= \rho_\theta(\eta_0)\rho_\theta(\eta_1) a\theta^{-1} && (\text{since } \delta < \beta)\\ &= \rho_\theta(\eta_0\eta_1) a\theta^{-1}\\ &= (\eta_0\eta_1) . a\theta^{-1}\end{aligned}$$

the latter since the order of the standard form of $\eta_0\eta_1$ is less than β by Lemma 1.1.

Case 2 ($\delta < \gamma = \beta$)

$$\begin{aligned}\eta_0 . (\eta_1 . a\theta^{-1}) &= \eta_0 . (\hat{\eta}_1 . aX_\beta^{m-n}\hat{\theta}^{-1}) && (\text{clause 2 of the definition})\\ &= (\eta_0\hat{\eta}_1) . aX_\beta^{m-n}\hat{\theta}^{-1} && (\text{induction hypothesis})\end{aligned}$$

$$\begin{aligned}(\eta_0\eta_1) . a\theta^{-1} &= (\eta_0\hat{\eta}_1 x_\beta^m) . a\theta^{-1}\\ &= (\eta_0\hat{\eta}_1) . aX_\beta^{m-n}\hat{\theta}^{-1} && (\text{clause 2 of the definition})\end{aligned}$$

the latter since the order of the standard form of $\eta_0\hat{\eta}_1$ is less than β.

Case 3 ($\gamma < \delta = \beta$)

$$\begin{aligned}\eta_0 . (\eta_1 . a\theta^{-1}) &= \eta_0 . \rho_\theta(\eta_1) a\theta^{-1} && (\text{since } \gamma < \beta)\\ &= \hat{\eta}_0 . \rho_\theta(\eta_1) aX_\beta^{l-n}\hat{\theta}^{-1} && (\text{clause 2 of the definition})\end{aligned}$$

$$\begin{aligned}(\eta_0\eta_1) . a\theta^{-1} &= (\hat{\eta}_0 x_\beta^l \eta_1) . a\theta^{-1}\\ &= (\hat{\eta}_0 \rho_\beta^l(\eta_1) x_\beta^l) . a\theta^{-1} && (\text{since } \gamma < \beta)\\ &= \hat{\eta}_0 . (\rho_\beta^l(\eta_1) x_\beta^l . a\theta^{-1}) && (\text{induction hypothesis})\\ &= \hat{\eta}_0 . (\rho_\beta^l(\eta_1) . aX_\beta^{l-n}\hat{\theta}^{-1}) && (\text{clause 2 of the definition})\\ &= \hat{\eta}_0 . \rho_{\hat{\theta}}\Big(\rho_\beta^{n-l}(\rho_\beta^l(\eta_1))\Big) aX_\beta^{l-n}\hat{\theta}^{-1} && (\text{clause 1 of the definition})\\ &= \hat{\eta}_0 . \rho_\theta(\eta_1) aX_\beta^{l-n}\hat{\theta}^{-1}\end{aligned}$$

Case 4 ($\delta = \gamma = \beta$)

$$\begin{aligned}\eta_0 . (\eta_1 . a\theta^{-1}) &= \eta_0 . (\hat{\eta}_1 . aX_\beta^{m-n}\hat{\theta}^{-1}) \quad \text{(clause 2 of the definition)}\\ &= (\eta_0\hat{\eta}_1) . aX_\beta^{m-n}\hat{\theta}^{-1} \quad \text{(induction hypothesis)}\\ &= \rho_{\eta_0}(\hat{\eta}_1)\eta_0 . aX_\beta^{m-n}\hat{\theta}^{-1} \quad (\text{since } \operatorname{ord}(\hat{\eta}_1) < \operatorname{ord}(\eta_0))\\ &= \rho_{\eta_0}(\hat{\eta}_1)\hat{\eta}_0 . aX_\beta^{l+m-n}\hat{\theta}^{-1} \quad \text{(clause 2 of the definition)}\end{aligned}$$

$$\begin{aligned}(\eta_0\eta_1) . a\theta^{-1} &= (\eta_0\hat{\eta}_1 x_\beta^m) . a\theta^{-1}\\ &= (\rho_{\eta_0}(\hat{\eta}_1)\eta_0 x_\beta^m) . a\theta^{-1} \quad (\text{since } \operatorname{ord}(\hat{\eta}_1) < \operatorname{ord}(\eta_0))\\ &= (\rho_{\eta_0}(\hat{\eta}_1)\hat{\eta}_0 x_\beta^{l+m}) . a\theta^{-1}\\ &= \rho_{\eta_0}(\hat{\eta}_1)\hat{\eta}_0 . aX_\beta^{l+m-n}\hat{\theta}^{-1} \quad \text{(clause 2 of the definition)}\end{aligned}$$

■

Now I begin to consider the task of solving linear equations in E_0. In particular I will show that for $s \in S$ and for any $e \in E_0$, the equation $s.v = e$ has a unique solution in E_0. Thus E_0 is in fact an $\mathcal{R}^\alpha$-module. I will also develop the tools for finding all solutions to division problems.

Lemma 4.6. *Scalar multiplication cannot increase the degree. More precisely, let* $\eta, \theta \in \Theta$. *Then*

1. $\deg(\eta . \theta^{-1}) \leq \deg(\theta^{-1})$.
2. If $\operatorname{ord}(\eta) = -1$ *or* $\operatorname{ord}(\eta) < \operatorname{ord}(\theta)$ *then* $\deg(\eta . \theta^{-1}) = \deg(\theta^{-1})$.
3. If $\operatorname{ord}(\eta) \geq 0$ *and* $\operatorname{ord}(\eta) \geq \operatorname{ord}(\theta)$ *then* $\deg(\eta . \theta^{-1}) < \deg(\theta^{-1})$.
4. If $\deg(\eta) > \deg(\theta)$, *then* $\eta . \theta^{-1} = 0$.

Proof. Immediate by the definition of multiplication. ■

Lemma 4.7. *For any* $s \in S$ *the unique solution to* $s.v = 0$ *in* E_0 *is* $v = 0$.

Proof. Since s has a non-zero constant term s_0, it follows from Lemma 4.6 that if e_0 is the term of e of largest degree, then $s_0 . e_0$ is the term of $s . e$ of largest degree; that is, if $e \neq 0$ then $s . e \neq 0$. ■

Lemma 4.8. *If* $0 \neq e$ *with* $a_0\theta_0^{-1}$ *being the term of e of largest degree, then* $\operatorname{ann}(e) = \langle\langle x_0\theta_0 \rangle\rangle$.

Proof. If $a\theta^{-1}$ is any other term of e then already $\theta_0 . a\theta^{-1} = 0$ by Lemma 4.6. Clearly $x_0\theta_0 . a\theta_0^{-1} = 0$. On the other hand, if $r . e = 0$ we can write r in the form $s\theta_1$ with $s \in S$ and $\theta_1 \in \Theta$. Then $s\theta_1 . e = 0$ implies that $\theta_1 . e = 0$ by Lemma 4.7. Clearly if $\theta_1 \preceq \theta_0$ then $\theta_1 . a\theta_0^{-1} \neq 0$ by the definition of multiplication; and so by Lemma 4.6 $\theta_1 . e \neq 0$. Thus $\theta_1 \succ \theta_0$, that is, $\theta_1 \in \langle\langle x_0\theta_0 \rangle\rangle$. ■

Lemma 4.9. *For $\eta \in \Theta$, the solutions in E_0 to the equation $\eta . v = 0$ are exactly the elements of $\sum_{\theta<\eta} K_0\theta^{-1}$, that is, the elements of lesser degree than η.*

Proof. Obvious. ■

Lemma 4.10. *To solve all equations of the form $r . v = e$ $(r \in R^\alpha, e \in E_0)$ in E_0 it suffices to be able to solve all equations of the forms $s . v = a\theta^{-1}$ and $\eta . v = a\theta^{-1}$, where $s \in S$, $\theta \in \Theta$, $\eta \in \Theta$, and the situations $a = 0$, $\theta = 1$ are allowed.*

Proof. If $e \neq 0$, express e as a sum of terms $e = \sum_{i<n} a_i\theta_i^{-1}$. Clearly if we have solutions to each equation $r . v = a_i\theta_i^{-1}$ then we get a solution to the original equation as a sum of these. Any $r \in R^\alpha$ can be written in the form $r = s\eta$ where $s \in S$ and $\eta \in \Theta$. Thus $r . v = e$ if and only if $s . (\eta . v) = e$. So if we can solve $s . w = e$ and $\eta . v = w$, then we can solve $r . v = e$. Combining the two reductions yields the Lemma. ■

Lemma 4.11. *Every equation of the form $\eta . v = a\theta^{-1}$, $\eta \in \Theta$, has a solution in E_0; in particular a canonical solution (when $a \neq 0$) is found as follows: Write $a\theta^{-1} = aX_\gamma^{-n}\hat{\theta}^{-1}$ and split up the standard form of η as $\eta = \eta_0 x_\gamma^m \eta_1$ (with any of $\eta_0 = 1$, $m = 0$ and $\eta_1 = 1$ allowed). Then $v = [\rho_\theta(\eta_0)]^{-1} a\eta_1^{-1} X_\gamma^{-n-m}\hat{\theta}^{-1}$ is a solution.*

Proof. Since $a\theta^{-1}$ is a solution to $\theta . v = a$, $\eta . v = a\theta^{-1}$ implies that $(\theta\eta) . v = a$. Simple computations from the definition of multiplication give the standard form of $(\theta\eta)^{-1}$ as above. ■

Note that the formula given simplifies considerably if $\mathrm{ord}(\eta) < \mathrm{ord}(\theta)$: a solution is $v = [\rho_\theta(\eta)]^{-1} a\theta^{-1}$. We get all solutions to $\eta . v = a\theta^{-1}$ by combining the above with Lemma 4.9. Note that the degree of any solution is the degree of $\theta\eta$.

Lemma 4.12. *Every equation of the form $s . v = a\theta^{-1}$, $s \in S$, has a solution in E_0, and the degree of the solution is* $\deg(\theta)$.

Proof. I prove this by induction on $\deg(\theta)$. If $\deg(\theta) = 0$ then $a\theta^{-1} = a \in K$; write $s = s_0 + s_1$ where $s_0 \in K$ and $s_1 \notin S$; then $s.v = a$ has the solution $v = s_0^{-1}a$.

So assume that $\deg(\theta) \geq 1$ (so that $\mathrm{ord}(\theta) \geq 0$) and that $s . v = a\theta_0^{-1}$ has a solution in E_0 for all $s \in S$ and all $a\theta_0$, $\theta_0 \in \Theta$, of degree less than $\deg(\theta)$. Write $s = s_0 + s_1$ where s_0 is the sum of all terms of s of order less than $\mathrm{ord}(\theta)$; hence every term of s_1 has order $\geq \mathrm{ord}(\theta)$ or $s_1 = 0$. Clearly $s_0 \in S$. Let $c = [\rho_\theta(s_0)]^{-1}a$. Let $e = a\theta^{-1} - s . c\theta^{-1}$. Then

$e = a\theta^{-1} - s_0 . c\theta^{-1} - s_1 . c\theta^{-1} = a\theta^{-1} - \rho_\theta(s_0)c\theta^{-1} - s_1 . c\theta^{-1}$ (since the order of each term of s_0 is less than the order of θ^{-1}), and this is then equal to $a\theta^{-1} - a\theta^{-1} - s_1 . c\theta^{-1} = -s_1 . c\theta^{-1}$. Notice that if $s_1 = 0$ then we are done. Otherwise, the order of each term of s_1 is greater than the order of θ_0 so by Lemma 4.6 the degree of each term of $e = -s_1 . c\theta^{-1}$ is less than the degree of θ. Hence by Lemma 4.10 and the induction hypothesis, the equation $s . w = e$ has a solution e_0 in E_0, of degree less than $\deg(\theta)$. Then $s . (c\theta^{-1} + e_0) = s . c\theta^{-1} + s . e_0 = s . c\theta^{-1} + e = s . c\theta^{-1} + a\theta^{-1} - s . c\theta^{-1} = a\theta^{-1}$. Clearly $\deg(c\theta^{-1} + e_0) = \deg(\theta)$. ■

Note that the proofs of the preceding lemmas actually give an algorithm for solving any equation $r . v = e$ in the R^α-module E_0. However the obvious restrictions on the allowable degrees of terms in the answer give an easier method of computing solutions. Write down a formal sum with undetermined coefficients of the finitely many monomials that could actually appear in a solution and are not annihilated by r; multiply formally by r; solve by comparing coefficients to the representation of e as a inverse polynomial. Unfortunately this more efficient method of solving equations does not yield an efficient proof of the desired result following. For to use this method to prove the main theorem, we would be need to be able to show that the systems of linear equations over K arising by comparing coefficients are always consistent.

Corollary 4.13. *E_0 is a divisible $\mathcal{R}^\alpha$-module.* ■

Now it is trivial to see that E_0 is an essential extension, and hence the injective envelope, of $\mathcal{R}^\alpha/J$.

Lemma 4.14. *E_0 is an essential extension of $\mathcal{R}^\alpha/J$.*

Proof. Let $0 \neq e \in E_0$ and let $k_0\theta_0^{-1}$ be the term of e of highest degree. Then if $k\theta^{-1}$ is any other term of e, $\theta_0 . k\theta^{-1} = 0$. Hence $\theta_0 . e = \theta_0 . k_0\theta_0^{-1} = k_0$, a non-zero element of $\mathcal{R}^\alpha/J$. ■

Theorem 4.15. *E_0 is the injective envelope of $\mathcal{R}^\alpha/J$.* ■

Exercise 4.16. Find all solutions to the following system of linear equations over $\mathrm{E}(\mathcal{R}^\alpha/J)$.

$$x_0 . u + x_1 . v = X_1^{-1}$$
$$u + (x_0 + x_1) . v = 1 + X_1^{-1}$$

A detailed solution is available from the author.

4.2 All the Other Indecomposable Injectives

Now the descriptions of all the $E(\mathcal{R}^\alpha/\mathcal{R}^\alpha x_\beta)$, $0 < \beta < \alpha$, follow in a similar manner. The reader should note the similarities to the commutative ring constructions in[6], in particular the appearance in the descriptions of the quotient fields K^β, taken over successively larger subdomains of $\mathcal{R}^\alpha$. This is an indication that in spite of the many peculiarities exhibited by both the ring and its indecomposable injectives, the $E(\mathcal{R}^\alpha/\mathcal{R}^\alpha x_\beta)$ are in many ways quite well behaved injective modules.

First note that since $\mathcal{R}^\alpha$ is an Ore domain, it has a quotient skew field K^α, and this quotient field is, as a left $\mathcal{R}^\alpha$-module, the injective envelope of $\mathcal{R}^\alpha$.

Theorem 4.17. *The injective envelope of $\mathcal{R}^\alpha$ is the quotient skew field of $\mathcal{R}^\alpha$.*

Proof. K^α is clearly an essential extension of $\mathcal{R}^\alpha$, and as already noted, it suffices to check divisibility. But this is obvious. ■

So for the moment, I let β be any ordinal, $0 < \beta < \alpha$. I begin, as in the previous section, by considering the effects of monomial scalar multiplication on the solutions to monomial equations.

Note that since K^β is clearly an essential extension of its $\mathcal{R}^\alpha$-submodule $\mathcal{R}^\alpha/\mathcal{R}^\alpha x_\beta$, we might as well asssume that the constants of these equations are in K^β. So consider an equation $\theta . v = a$ with $\theta \in \Theta$ and $a \in K^\beta$. Split the standard monomial θ into two (possibly empty) parts, $\theta = \theta_0\theta_1$ with $\theta_0 \in \bar{R}_\beta$ and $\theta_1 \in R^\alpha x_\beta$. Then any solution to $\theta . v = a$ is also a solution to $\theta_1 . v = (1/\theta_0)a$ and conversely. So we might as well assume that $\theta \succeq x_\beta$. (The reader should be careful in the sequel about the distinction between reciprocals in K^β such as $(1/\theta_0)$ above, and the formal inverse monomials that will be introduced later; in this context θ_1^{-1} will be one such.)

Lemma 4.18. *Let θ and η be standard monomials, $\theta \succeq x_\beta$, and let $a \in K^\beta$. Let e be any solution in $E(\mathcal{R}^\alpha/\mathcal{R}^\alpha x_\beta)$ to $\theta . v = a$. Then:*

1. $\eta . a = \eta a$ if $\eta \prec x_\beta$ *and* $\eta . a = 0$ if $\eta \succeq x_\beta$.
2. *If* $\mathrm{ord}(\eta) < \mathrm{ord}(\theta)$ *then* $\eta . e$ *is a solution to* $\theta . v = \rho_\theta(\eta)a$.
3. *If* $\mathrm{ord}(\eta) = \mathrm{ord}(\theta)$, *(and so* $\eta \succeq x_\beta$*), say* $\eta = \hat{\eta}x_\gamma^m$ *and* $\theta = \hat{\theta}x_\gamma^n$, *and* $m \leq n$ *then* $\eta . e = \hat{\eta} . \hat{e}$, *where* $\hat{e}$ *is a solution to* $\hat{\theta}x_\gamma^{n-m} . v = a$.
4. *If* $\mathrm{ord}(\eta) = \mathrm{ord}(\theta)$ *as in (3) and* $m > n$ *or if* $\mathrm{ord}(\eta) > \mathrm{ord}(\theta)$, *(and so* $\mathrm{ord}(\eta) > \mathrm{ord}(x_\beta)$*), then* $\eta . e = 0$.

Proof. The same as the proof of lemma 4.1. ■

Definition 4.19. *Let* $\Theta_\beta = \{\theta \in \Theta : \text{no } x_\gamma, \gamma < \beta \text{ occurs in } \theta\}$ *(and so* $1 \in \Theta_\beta$*). For* $\theta \in \Theta_\beta$, *let* K_θ^β *be the* K^β*-vector space on* K^β *with scalar action* $t^{-1}r . k = [\rho_\theta(t)]^{-1}\rho_\theta(r)k$, *for* $0 \neq t$, $r \in \bar{R}_\beta$.

Let $E_\beta = \oplus_{\theta \in \Theta_\beta} K_\theta^\beta \theta^{-1}$.
Define an action of R^α *on* E_β *as before:*

$$\eta \,.\, a\theta^{-1} = \begin{cases} \rho_\theta(\eta) a\theta^{-1} & \mathrm{ord}(\eta) < \mathrm{ord}(\theta)\,, \\ \hat{\eta} \,.\, aX_\beta^{m-n}\hat{\theta}^{-1} & \mathrm{ord}(\eta) = \mathrm{ord}(\theta)\,, \eta = \hat{\eta}x_\gamma^m \text{ and } \theta = \hat{\theta}x_\gamma^n\,, m \le n\,, \\ 0 & \mathrm{ord}(\eta) = \mathrm{ord}(\theta)\,, m > n\,, \text{ or } \mathrm{ord}(\eta) > \mathrm{ord}(\theta)\,. \end{cases}$$

Lemma 4.20.

1. E_β *is a left* R^α*-module.*
2. E_β *is a divisible left* $\mathcal{R}^\alpha$*-module.*
3. E_β *is an essential extension of* K^β.

Proof. These follow for exactly the same reasons as before. ■

Theorem 4.21. E_β *is the injective envelope of* $\mathcal{R}^\alpha/\mathcal{R}^\alpha x_\beta$. ■

Note that if I had allowed $\beta = \alpha$ above, we would have $\Theta_\beta = \{1\}$, so naming $\mathrm{E}(\mathcal{R}^\alpha)$ as E_α is consistent with Definition 4.19.

4.3 Applications

For the most part, I leave it to the reader to explore possible applications of these descriptions. Computational problems such as solving systems of linear equations, while daunting, are feasible. Extracting more useful algebraic (or model-theoretic) information is of course more of a challenge. In these examples once again, the (algebraic) socle series stabilizes after only ω steps, whereas the elementary socle series quite naturally corresponds to the structure of the description.

Proposition 4.22 (socle series).

1. $\mathrm{soc}_n(E_0) = \sum_{m<n} K_{x_0^m} X_0^{-m}$.
2. *For all* $\gamma \ge \omega$, $\mathrm{soc}_\gamma(E_0) = \sum_{m<\omega} K_{x_0^m} X_0^{-m}$.

Proof. The key point is that $\rho_0 : K \to K$ is an embedding of a field into K, whereas for $\beta > 0$, $\rho_\beta : \bar{R}_\beta \to K$ is an embedding of a domain not a field into K. So, for instance, the minimal submodules over $\mathrm{soc}_1(E_0) = K$ are of the form $K + \rho_0[K]aX_0^{-1}$ for some $a \in K$. For $b \in \rho_0[K]a$ if and only if $b = da$ for some $d \in \rho_0[K]$; and since $\rho_0[K]$ is a subfield, $d^{-1} \in \rho_0[K]$ so $d^{-1}b = a$ and $a \in \rho_0[K]b$. On the other hand, $(\rho_1(x_0))^{-1}$ is not in the image of ρ_1 and so $\langle\langle X_1^{-1}\rangle\rangle$, $\langle\langle \rho_1(x_0)X_1^{-1}\rangle\rangle$, $\langle\langle \rho_1(x_0^2)X_1^{-1}\rangle\rangle$, $\cdots$ form an infinite descending chain of submodules with intersection K, but no one of them is contained in $\mathrm{soc}_\omega(E_0)$. ■

Proposition 4.23 (elementary socle series). $\mathrm{soc}^{\delta}(E_0) = \sum_{\theta,\deg(\theta)<\delta} K_\theta \theta^{-1}$ *for* $\delta \leq \omega^{\alpha}$.

In particular, for $\gamma < \alpha$, $\mathrm{soc}^{\omega^{\gamma}}(E_0) = \sum_{\theta,\theta \prec x_\gamma} K_\theta \theta^{-1}$.

Proof. The definable subgroups are determined by Lemma 4.9. ■

I will investigate only one application of substance. We saw in Sec. 2 that the "uncomplicated" indecomposable $(\mathrm{E}(R/J^{\omega}))$ was a homomorphic image of the "complicated" one $(\mathrm{E}(R/J))$. This "layering" of the indecomposables is one of the interesting and peculiar features of the example. This sort of structure is displayed beautifully in the general case.

Theorem 4.24. *Let* $0 < \beta < \alpha$. *Then for each* γ, $\beta \leq \gamma < \alpha$, $E_0/\mathrm{soc}^{\omega^{\beta}}(E_0)$ *has an infinite direct sum of copies of* E_γ *as a direct summand.*

Proof. First note that $A_\beta = \sum_{\theta,\theta \succeq x_\beta} K_\theta \theta^{-1}$ is a set of representatives of the cosets of $\mathrm{soc}^{\omega^{\beta}}(E_0)$ in E_0. Scalar multiplication on A_β is then just ordinary scalar multiplication on E_0, setting to 0 any term of degree less than $\deg(x_\beta)$. Since $\mathcal{R}^{\alpha}$ is hereditary, A_β is injective. So it suffices to find linearly independent elements of A_β with annihilator $\mathcal{R}^{\alpha}x_\gamma$ for any γ, $\beta \leq \gamma < \alpha$.

In fact, for fixed γ, the set $\{ X_\gamma^{-1}\theta^{-1} : \theta \prec x_\beta \}$ will do. For in E_0, $r\,.\,X_\gamma^{-1}\theta^{-1}$ either has order that of x_γ, or order less that that of x_β. So in A_β, $r\,.\,X_\gamma^{-1}\theta^{-1} = 0$ if and only if $r \in \mathcal{R}^{\alpha}x_\gamma$. Furthermore, if $r\,.\,X_\gamma^{-1}\theta^{-1}$ is not zero in A_β, the product retains a term in $X_\gamma^{-1}\theta^{-1}$ in A_β. So no non-trivial linear combination of such things can be zero. For the same reason, the elements chosen for different γ's are independent from each other. ■

Something wrong with an inequality here (\beta)

I have avoided giving the complete decomposition of $E_0/\mathrm{soc}^{\omega^{\beta}}(E_0)$ as a direct sum of indecomposables. This is in part due to difficulties presented by the fine details of the construction of Jategaonkar's ring $\mathcal{R}^{\alpha}$. Clearly the map ρ_β extends to an embedding of K^{β} into K. This makes K into a left vector space over K^{β}. What is the dimension of this space? It is clear that if $\{ a_i : i \in I \}$ is a basis for K over K^{β} then $\{ a_i X_\beta^{-1} : i \in I \}$ is also a linearly independent set of elements of A_β with annihilator $\mathcal{R}^{\alpha}x_\beta$. So the determination of the actual multiplicity of E_β in A_β (or of any E_γ, $\gamma \geq \beta$) depends on specific details of the construction of $\mathcal{R}^{\alpha}$. However it appears that with the appropriate choices in the initial conditions of the construction of $\mathcal{R}^{\alpha}$, we can guarantee that the multiplicity of each E_γ, $\beta \leq \gamma \leq \alpha$, in A_β is $|\mathcal{R}^{\alpha}|$. Of course, results similar to Theorem 4.24 hold for each E_β.

ACKNOWLEDGMENTS

This paper has had a rather long gestation, and I have benefited from the kind advice of many people over the last six years. This work began in

1994 with the main result of Sec. 2 while I was spending a portion of my sabbatical at the University of Notre Dame; I would like to thank that institution for its hospitality and in particular Anand Pillay. Most of the basic ideas in the rest of the paper were worked out during visits to the University of Manchester and the University of Leeds in 1996 and 1997. I would like to thank Mike Prest, Chris Robson, John McConnell, and Dugald MacPherson for their patience and attention. My understanding of the rings $\mathcal{R}^\alpha$ was greatly helped by conversations with Jan Trlifaj in Manchester; and the research was finished during a visit with him at Charles University in the summer of 1998. I have included Sec. 3.4 in partial response to some of his questions.

Research supported by a grant from the Natural Sciences and Engineering Research Council of Canada.

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Received August 2001