Explicit descriptions of the indecomposable injective modules over Jategaonkar's rings Supplement to Theorem 4.24 *†

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Abstract

In 1968–1969, A. V. Jategaonkar published his famous constructions of left but not right noetherian rings that provided counterexamples to several important conjectures of that era. These examples, and others like them, seemed to indicate that, in general, the task of completely understanding the structure of indecomposable injective modules over one-sided noetherian rings was hopeless. In this paper I show how to deduce by natural methods, directly from the known description of these rings and their properties, explicit computational descriptions of the indecomposable injective left modules over Jategaonkar's rings. I use these explicit descriptions to answer some simple structural questions about the indecomposables.

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The proof outline given of Theorem 4.24 in the original is entirely opaque and certainly includes at least some errors in inequalities. A correct proof is given, together with additional material that can be extracted from the structural description.

I have left a lot of "empty" items in, in order to preserve numbering and cross-references.

1 Descriptions of the rings

1.1 The ring R - a simple case

1.2 The rings $\mathcal{R}^\alpha$ — the general case.

Now I describe the more complicated rings from [3]. It would be nice to follow Jategaonkar's description exactly, but unfortunately a choice that he makes in order to make the exposition of his construction simpler (starting the enumeration of indeterminates with x_1 rather than with x_0) would make the description of my constructions unwieldy, with separate cases for finite and infinite β. So with the minor change in the indexing of indeterminates, I present Jategaonkar's examples.

In [3, see especially Theorem 4.6], Jategaonkar shows the existence of rings R^α, α an ordinal, with the following description and properties.

There is a division ring $K \subset R^\alpha$ and twisted polynomial extensions of K inside R^α satisfying:

$$\begin{array}{rcll} \bar{R}_\beta &=& K[x_\gamma, \rho_\gamma : \gamma < \beta] & \text{for each } \beta \leq \alpha\,. \\ R_\beta &=& \bar{R}_\beta[x_\beta, \rho_\beta] & \text{for each } \beta < \alpha\,. \\ R^\alpha &=& \bar{R}_\alpha\,. & \end{array}$$

where each $\rho_\beta \colon \bar{R}_\beta \longrightarrow K$ is a monomorphism, and multiplication in R^α is determined by $x_\beta r = \rho_\beta(r) x_\beta$ for any r in $\bar{R}_\beta$.

Elements of R^α can be expressed in an essentially unique way as a (finite) sum of distinct *standard monomials*, where a standard monomial is a term of the form $a\, x_{\alpha_1}^{n_1} \ldots x_{\alpha_k}^{n_k}$ for some $k \geq 0$, $a \in K$, $\alpha_1 < \cdots < \alpha_k < \alpha$, and $n_i > 0$. (For $k = 0$, a standard monomial is just some element of K.) Jategaonkar shows that R^α is a principal left ideal domain and that the elements $1+x_\beta$, $\beta < \alpha$, are right R^α-linearly independent. From these facts follow most of the interesting and peculiar properties of the ring R^α.

Let Θ be the set of all monic standard monomials (including 1). The set Θ can be ordered in order type ω^α as follows. The least element is 1, and otherwise given two monic standard monomials, write them with common variables as $\theta_1 = x_{\alpha_1}^{n_1} \ldots x_{\alpha_k}^{n_k}$ and $\theta_2 = x_{\alpha_1}^{m_1} \ldots x_{\alpha_k}^{m_k}$, $\alpha_1 < \cdots < \alpha_k < \alpha$ and $n_i \geq 0$, $m_i \geq 0$. Then $\theta_1 \prec \theta_2$ if and only if for some t, $n_l = m_l$ for all l, $t < l \leq k$, and $n_t < m_t$. Consequently, each standard monomial has a well defined degree which is an ordinal $< \omega^\alpha$, and I define the degree of a non-zero element of R^α to be the maximum of the degrees of its terms. Note in particular that $\deg(1) = 0$ and that $\deg(x_\beta) = \omega^\beta$. Note also that any subsequence of a monic standard polynomial is again a member of Θ, in particular initial and final segments (including the empty segment, taken to be 1) of some $\theta \in \Theta$ are again in Θ. For any $\theta \in \Theta$ and $a \in K$, I define the *order* of $a\theta$ to be -1 if $\theta = 1$, otherwise it is the largest index of a variable occurring in θ. For $\theta \in \Theta$, $\theta = x_{\alpha_1}^{n_1} \ldots x_{\alpha_k}^{n_k}$, I let $\rho_\theta = \rho_{\alpha_1}^{n_1} \circ \cdots \circ \rho_{\alpha_k}^{n_k}$, with $\rho_\emptyset$ the identity map. I record a few simple but useful facts about degree and order.

Lemma 1.1 *1. The successor of θ in the order on Θ is $x_0\theta$.*

2. Let $0 \neq r$, $s \in R^\alpha$. Then[1]

$$\deg(rs) = \deg(s) + \deg(r)\,.$$

3. Let θ_0, θ_1 be standard monomials. Then

$$\mathrm{ord}(\theta_0\theta_1) = \max\{\mathrm{ord}(\theta_0), \mathrm{ord}(\theta_1)\}\,.$$

4. Let θ_0, θ be standard monomials. If $\mathrm{ord}(\theta_0) < \mathrm{ord}(\theta)$ then $\theta\theta_0 = \rho_\theta(\theta_0)\theta$.

Proof: The first three parts are obvious. The last part is proved by induction on the order of θ. If $\theta \in K$ there is nothing to prove. Otherwise $\theta = \theta' x_\beta^n$ with $\mathrm{ord}(\theta_0) < \beta$. Then $\theta\theta_0 = \theta' x_\beta^n \theta_0 = \theta' \rho_\beta^n(\theta_0) x_\beta^n$ (since $\mathrm{ord}(\theta_0) < \beta$, θ_0 is in the domain of ρ_β), and then since $\rho_\beta^n(\theta_0) \in K$, by induction hypothesis the latter equals $\rho_{\theta'}(\rho_\beta^n(\theta_0))\theta' x_\beta^n$, that is, it equals $\rho_\theta(\theta_0)\theta$. ∎

[1] Here '+' represents ordinal addition; the order of r and s on both sides is significant, as neither operation is commutative.

Let $S = \{ f \in R^\alpha : f$ has non zero constant term $\}$. S is easily seen to be a left Ore set in R^α, [3, Theorem 4.5], and every non-zero element r of R^α can be written in the form $r = s\theta$, where $s \in S$ and $\theta \in \Theta$. In fact it follows immediately from the proof of [3, Theorem 4.5] that θ is just the indeterminate part of the term of r of least degree. Let $\mathcal{R}^\alpha = (R^\alpha)_S$. Every non-zero element of $\mathcal{R}^\alpha$ can be written as a sum of terms of the form $s^{-1}\theta$ where $s \in S$ and $\theta \in \Theta$.

$\mathcal{R}^\alpha$ is a principal left ideal domain, every left ideal is generated by a monic standard monomial, every left ideal is two sided, and the left ideals are well-ordered by reverse inclusion. Thus $\mathcal{R}^\alpha$ is left FBN and so the indecomposable injective left $\mathcal{R}^\alpha$-modules are in one-to-one correspondence with the prime ideals. A typical non zero left ideal is then $\mathcal{R}^\alpha\theta$, with $\theta \in \Theta$. Then $\mathcal{R}^\alpha\theta_0 \supset \mathcal{R}^\alpha\theta_1$ if and only if $\theta_0 \prec \theta_1$.

Considerably more can be read out of the analysis of the descending sequence of left ideals in the proof of [3, Theorem 4.6] than is stated in the conclusion of the theorem. In fact we have that the Jacobson radical $J(\mathcal{R}^\alpha)$ is the ideal $J = \mathcal{R}^\alpha x_0$, and every left ideal is J^β for some β, so J^β is the β-th ideal in the descending order just described. In particular, the prime ideals of $\mathcal{R}^\alpha$ are $J, J^\omega = \mathcal{R}^\alpha x_1, \dots, J^{\omega^n} = \mathcal{R}^\alpha x_n, \dots, J^{\omega^\omega} = \mathcal{R}^\alpha x_\omega, \dots, J^{\omega^\beta} = \mathcal{R}^\alpha x_\beta, \dots,\ J^{\omega^\alpha} = 0$. (Once again the same model-theoretic explanation as in Section 1.1 could be used to show that these are exactly the prime ideals.)

For the purpose of developing explicit descriptions of the indecomposable injectives, it will be useful to have descriptions of $\mathcal{R}^\alpha/\mathcal{R}^\alpha x_\beta$ for each $\beta < \alpha$. Let $S_\beta = S \cap \bar{R}_\beta$. Then S_β is a left Ore set in $\bar{R}_\beta$. Clearly for any $s \in S$ there is unique $\bar{s} \in S_\beta$ such that $s - \bar{s} \in R^\alpha x_\beta$; similarly for any $r \in R^\alpha$ there is a unique $\bar{r} \in \bar{R}_\beta$ such that $r - \bar{r} \in R^\alpha x_\beta$. Thus for any $s^{-1}r \in \mathcal{R}^\alpha$, $s^{-1}r \equiv \bar{s}^{-1}\bar{r}\ (\mathcal{R}^\alpha x_\beta)$. So $\mathcal{R}^\alpha/\mathcal{R}^\alpha x_\beta \cong (\bar{R}_\beta)_{S_\beta}$. Here the action of $\mathcal{R}^\alpha$ on the $\mathcal{R}^\alpha$-module $(\bar{R}_\beta)_{S_\beta}$ can be thought of as "ordinary multiplication in $\mathcal{R}^\alpha$, followed by setting to 0 any indeterminate x_γ, $\gamma \geq \beta$". Furthermore, since by construction $\bar{R}_\beta$ is a left Ore domain ([3, Theorem 2.8]), we may consider $(\bar{R}_\beta)_{S_\beta}$ to be embedded in the left quotient field K^β of $\bar{R}_\beta$.

2 The indecomposable injectives over R

Lemma 2.1 *Any injective right R-module E is also an injective right D-module and an injective right $F[x]$-module.*

Lemma 2.2 *If $(c_i)_{i\in I}$ is F-linearly independent in C, and for each $i \in I$,*

e_i *is a solution in* $\mathrm{E}(R/J)$ *to* $v \,.\, \langle 1,\, 0\rangle = c_i$ *, then* $(e_i)_{i\in I}$ *is* F*-linearly independent in* E *.*

Proof: Suppose that for some finite $I' \subset I$, $\sum_{i\in I'} e_i \,.\, \langle 0,\, a_i\rangle = 0$ where $a_i \in F$. Then $e_i \,.\, \langle 0,\, a_i\rangle \langle 1,\, 0\rangle = e_i \,.\, \langle 1,\, 0\rangle \langle 0,\, \sigma(a_i)\rangle = c_i \,.\, \langle 0,\, \sigma(a_i)\rangle = c_i\sigma(a_i)$. So certainly (in F), $\sum_{i\in I'} c_i\sigma(a_i) = 0$; thus in C_F , $\sum_{i\in I'} c_i \,.\, a_i = 0$. Thus by assumption, $a_i = 0$ for all $i \in I'$. ∎

Lemma 2.3 *Let* $(c_i)_{i\in I}$ *be an* F*-basis for* C_F *, and for each* $i \in I$ *, let* e_i *be a solution in* $\mathrm{E}(R/J)$ *of* $v \,.\, \langle 1,\, 0\rangle = c_i$ *. If* $a \in F$ *, and* $e \in E$ *is a solution of* $v \,.\, \langle 1,\, 0\rangle = a$ *, then* e *is an* F*-linear combination of the* e_i*'s modulo* $\{\, f \in E : f \,.\, J^n = 0 \text{ for some } n \,\}$ *.*

Theorem 2.4 $\mathrm{E}(R/J) \cong F[X^{-1}] \oplus X^{-\infty}F$ *with* R*-action given by*

$$(f(X^{-1}) + X^{-\infty}a) \,.\, \langle b,\, d\rangle = f(X^{-1}) \,.\, d + X^0ab + X^{-\infty}a\sigma(d)$$

(with the scalar action of D *on* $F[X^{-1}]$ *described earlier).*

Theorem 2.5 $\mathrm{E}(R/J^\omega) \cong F(x)$ *under the scalar action* $q \,.\, \langle b,\, d\rangle = qd$ *.* ∎

Note that $\mathrm{E}(R/J)/F[X^{-1}] \cong_R \mathrm{E}(R/J^\omega)$.

3 Some applications of the explicit description

3.1 Series in $\mathrm{E}(R/J)$

3.2 Endomorphism rings

Proposition 3.1 $\mathrm{End}(\mathrm{E}(R/J^\omega)) = F(x)$ *.* ∎

Lemma 3.2

$$\mathrm{End}_D(\mathrm{E}(R/J)) \cong \begin{pmatrix} F[[x]] & {}_{F[[x]]}F[x^{-1}, x]]_{F(x)} \\ 0 & F(x) \end{pmatrix}$$

with the actions on $F[x^{-1}, x]]$ *as described.* ∎

Quite a few computations have been left out of the above discussion, and the reader is warned that a detailed checking of these claims is, while routine, quite tedious.

Now we have to identify the R-endomorphisms among the D-endomorphisms. We represent an element of $\mathrm{E}(R/J)$ as a column vector with entries $f(X^{-1})$ and a, letting the endomorphism ring above act on these from the left by ordinary matrix multiplication. Again, equally routine and tedious computations yield:

Lemma 3.3 *The elements of* $\mathrm{End}_D(\mathrm{E}(R/J))$ *which are* R*-endomorphisms are exactly those of the form*

$$\begin{pmatrix} \xi & \zeta \\ 0 & \xi(0) \end{pmatrix} .$$

Theorem 3.4 *Let* $S = F[[x]] \oplus F[x^{-1}, x]]$ *with* $F[x^{-1}, x]]$ *the* $(F[[x]], F(x))$*-bimodule described earlier. Then* S *is a ring with multiplication*

$$\langle \xi, \zeta \rangle \langle \xi', \zeta' \rangle = \langle \xi\xi', \xi \centerdot \zeta' + \zeta \centerdot \xi'(0) \rangle \ .$$

S *is isomorphic to* $\mathrm{End}_R(\mathrm{E}(R/J))$*, acting on* $\mathrm{E}(R/J)$ *from the left according to the rule*

$$\langle \xi, \zeta \rangle \centerdot (f(X^{-1}) + X^{-\infty}a) = \xi \centerdot f(X^{-1}) + X^0\zeta \centerdot a + X^{-\infty}\xi(0)a \, .$$

∎

3.3 Elementary duals

Theorem 3.5 *1. The elementary dual of the right* R*-module* $\mathrm{E}(R/J)$ *is the flat indecomposable pure-injective left* R*-module* $P = F \oplus F[[x]]$*, with the action of* R *given by* $\langle b, d \rangle . \langle a, f \rangle = \langle b\rho(f) + \sigma(d)q, \eta(d)f \rangle$ *.*

2. The elementary dual of the right R*-module* $\mathrm{E}(R/J^\omega)$ *is the flat indecomposable pure-injective left* R*-module* $F(X)$*, with the action of* R *given by* $\langle b, d \rangle \, q = dq$ *.*

3.4 The lattice of submodules

Lemma 3.6 *$\mathcal{S}(\mathrm{E}(R/J^{\omega}))$ is isomorphic to the chain $1+\mathbb{Z}+1$.*

Lemma 3.7 *The following is a maximal chain in $\mathcal{S}(\mathrm{E}(R/J))$:*

$$0 \subset X^0F \subset \ldots \subset \sum_{i<n} X^{-i}F \subset \ldots \subset F[X^{-1}] \subset \ldots$$

$$\ldots \subset F[X^{-1}] + \langle\!\langle X^{-\infty}x_1^z \rangle\!\rangle \subset \ldots \subset \mathrm{E}(R/J)$$

where $n \in \omega$ and $z \in \mathbb{Z}$. ∎

Lemma 3.8 *For any $n \in \omega$ and $z \in \mathbb{Z}$, the following five submodules form a sublattice isomorphic to $\mathbf{M_3}$ in $\mathcal{S}(\mathrm{E}(R/J))$:*
$\langle\!\langle X^{-n}, X^{-\infty}x_1^z \rangle\!\rangle$, $\left\langle\!\left\langle X^{-(n+1)}, X^{-\infty}x_1^z \right\rangle\!\right\rangle$, $\left\langle\!\left\langle X^{-n}, X^{-(n+1)} + X^{-\infty}x_1^{z-1} \right\rangle\!\right\rangle$, $\left\langle\!\left\langle X^{-n}, X^{-\infty}x_1^{z-1} \right\rangle\!\right\rangle$, *and* $\left\langle\!\left\langle X^{-(n+1)}, X^{-\infty}x_1^{z-1} \right\rangle\!\right\rangle$. ∎

3.5 Solutions to the exercises

4 The indecomposable injectives over $\mathcal{R}^\alpha$

We have already seen in section 1.2 that the ring $\mathcal{R}^\alpha$ is left FBN and the prime ideals are exactly the ideals $J^{\omega^\beta} = \mathcal{R}^\alpha x_\beta$ for $\beta < \alpha$, and $J^{\omega^\alpha} = 0$, (where J is the Jacobson radical of $\mathcal{R}^\alpha$). Thus the indecomposable injectives of $\mathcal{R}^\alpha$ are precisely the injective envelopes $\mathrm{E}(\mathcal{R}^\alpha/J^{\omega^\beta})$, $\beta \leq \alpha$. By methods entirely similar to those outlined in detail in the previous section, we are led to explicit descriptions of these modules. Of course these also provide explicit descriptions of some indecomposable injective left R^α-modules, but it seems that it is unlikely that we would be able to identify in some uniform way *all* of the indecomposable injective left R^α-modules. Nonetheless, it seems reasonable that we should be able to describe explicitly any $\mathrm{E}_{R^\alpha}(R^\alpha/I)$ where we have reasonable descriptions of I and of R^α/I.

For each monic standard monomial $\theta \in \Theta$, $\theta = x_{\alpha_1}^{n_1} \ldots x_{\alpha_k}^{n_k}$, let θ^{-1} denote the formal expression $X_{\alpha_k}^{-n_k} \ldots X_{\alpha_1}^{-n_1}$ (an "inverse standard monomial"). (For $\theta = 1$ let θ^{-1} denote 1 as well.) The *order* of θ^{-1} is the order of θ. It is intended that in each indecomposable $\mathrm{E}(\mathcal{R}^\alpha/\mathcal{R}^\alpha x_\beta)$, $a\theta^{-1}$ will represent a canonical solution to a consistent equation $\theta \centerdot v = a$, where $\theta \geq x_\beta$ and $a \in K^\beta$.

The first (and major) task will be to describe $\mathrm{E}(\mathcal{R}^\alpha/J)$.

4.1 The "biggest" indecomposable $\mathrm{E}(\mathcal{R}^\alpha/J)$

Using the maps ρ_θ , $\theta \in \Theta$, K may be made into a vector space over itself since each ρ_θ restricts to a proper embedding of K into itself. Let K_θ denote the left K-vector space with underlying set K and scalar action $a \centerdot k = \rho_\theta(a)k$. Let E_0 be the K-vector space $\oplus_{\theta\in\Theta} K_\theta \theta^{-1}$. The elements of E_0 have a naturally defined degree inherited from the degree defined on R^α . I will show that E_0 is in fact the underlying K-vector space of $\mathrm{E}(\mathcal{R}^\alpha/J)$. Note that since $\mathcal{R}^\alpha$ is a domain, any equation $r \centerdot v = a$ with $a \in \mathcal{R}^\alpha/J$ and $r \neq 0$ is consistent.

Lemma 4.1 *Let θ and η be standard monomials, and let $a \in \mathcal{R}^\alpha/J \cong K$. Let e be any solution in $\mathrm{E}(\mathcal{R}^\alpha/J)$ to $\theta \centerdot v = a$. Then:*

1. *$\eta \centerdot a = \eta a$ if $\deg(\eta) = 0$ and $\eta \centerdot a = 0$ if $\deg(\eta) > 0$.*

2. *If $\mathrm{ord}(\eta) < \mathrm{ord}(\theta)$ then $\eta \centerdot e$ is a solution to $\theta \centerdot v = \rho_\theta(\eta)a$.*

3. *If $\mathrm{ord}(\eta) = \mathrm{ord}(\theta)$, say $\eta = \hat{\eta}x_\gamma^m$ and $\theta = \hat{\theta}x_\gamma^n$, and $m \leq n$ then $\eta \centerdot e = \hat{\eta} \centerdot \hat{e}$, where $\hat{e}$ is a solution to $\hat{\theta}x_\gamma^{n-m} \centerdot v = a$.*

4. *if $\mathrm{ord}(\eta) = \mathrm{ord}(\theta)$ as in (iii) and $m > n$ or if $\mathrm{ord}(\eta) > \mathrm{ord}(\theta)$, then $\eta \centerdot e = 0$.*

Proof:

1. $\eta \in K$ if $\deg(\eta) = 0$ and $\eta \in J$ if $\deg(\eta) > 0$.

2. Let $\eta \centerdot e = f$. Then $\theta \centerdot f = \theta \centerdot (\eta \centerdot e) = (\theta \centerdot \eta) \centerdot e = (\rho_\theta(\eta)\theta) \centerdot e$, by Lemma 1.1, which equals $\rho_\theta(\eta) \centerdot (\theta \centerdot e) = \rho_\theta(\eta) \centerdot a = \rho_\theta(\eta)a$, by part 1.

3. Let $\hat{e} = x_\gamma^m \centerdot e$. Then $\hat{\theta}x_\gamma^{n-m} \centerdot \hat{e} = \hat{\theta}x_\gamma^{n-m} \centerdot (x_\gamma^m \centerdot e) = (\hat{\theta}x_\gamma^{n-m}x_\gamma^m) \centerdot e = \theta \centerdot e = a$, and $\eta \centerdot e = \hat{\eta}x_\gamma^m \centerdot e = \hat{\eta} \centerdot (x_\gamma^m \centerdot e) = \hat{\eta} \centerdot \hat{e}$.

4. If $\mathrm{ord}(\eta) = \mathrm{ord}(\theta)$ and $m > n$ then $\eta \centerdot e = \hat{\eta}x_\gamma^m \centerdot e = \hat{\eta}x_\gamma^{m-n} \centerdot (x_\gamma^n \centerdot e)$, and $x_\gamma^n \centerdot e$ is a solution to $\hat{\theta} \centerdot v = a$ with $\mathrm{ord}(\hat{\theta}) < \mathrm{ord}(\hat{\eta}x_\gamma^{m-n})$, so we can assume without loss of generality that $\mathrm{ord}(\eta) > \mathrm{ord}(\theta)$. Thus $\eta\theta = \rho_\eta(\theta)\eta$ by Lemma 1.1. Let $f = \eta \centerdot e$. Then $\rho_\eta(\theta) \centerdot f = \rho_\eta(\theta) \centerdot (\eta \centerdot e) = (\rho_\eta(\theta)\eta) \centerdot e = (\eta\theta) \centerdot e = \eta \centerdot (\theta \centerdot e) = \eta \centerdot a = 0$, by part 1. But $\rho_\eta(\theta)$ is invertible, so $f = 0$.

$\blacksquare$

Corollary 4.2 *If* $\mathrm{ord}(\eta) = \mathrm{ord}(\theta)$ *and* $m < n$ *then* $\eta \centerdot e$ *is a solution to* $\hat{\theta}x_\gamma^{n-m} \centerdot v = \rho_{\hat{\theta}}(\rho_\gamma^{n-m}(\hat{\eta}))a$.

Proof: By part 3 and part 2, since if $m < n$ then $\mathrm{ord}(\hat{\eta}) < \mathrm{ord}(\hat{\theta}x_\gamma^{n-m})$. $\blacksquare$

With the intent that $a\theta^{-1}$ is supposed to represent a canonical solution to the (consistent) equation $\theta \centerdot v = a$, I define an action of the standard monomials on the inverse standard monomials by recursion on the order of η .

Definition 4.3

$$\eta \centerdot a\theta^{-1} = \begin{cases} \rho_\theta(\eta)a\theta^{-1} & \mathrm{ord}(\eta) = -1 \text{ } \textit{or} \text{ } \mathrm{ord}(\eta) < \mathrm{ord}(\theta)\,, \\ \hat{\eta} \centerdot aX_\gamma^{m-n}\hat{\theta}^{-1} & \mathrm{ord}(\eta) = \mathrm{ord}(\theta)\,, \eta = \hat{\eta}x_\gamma^m \text{ } \textit{and} \text{ } \theta = \hat{\theta}x_\gamma^n\,, m \leq n\,, \\ 0 & \mathrm{ord}(\eta) = \mathrm{ord}(\theta)\,, m > n\,, \text{ } \textit{or} \text{ } \mathrm{ord}(\eta) > \mathrm{ord}(\theta)\,. \end{cases}$$

Then I extend this to an action of R^α on E_0 by distributivity. It follows immediately from Lemma 4.1 that this multiplication respects the intended interpretation of the inverse monomials.

Corollary 4.4 *In the second case of the definition, if* $\mathrm{ord}(\eta) = \mathrm{ord}(\theta)$ *and* $m < n$ *then*

$$\eta \centerdot a\theta^{-1} = \rho_{\hat{\theta}}(\rho_\gamma^{n-m}(\hat{\eta}))aX_\gamma^{m-n}\hat{\theta}^{-1}\,.$$

$\blacksquare$

Note that the condition "$\mathrm{ord}(\eta) = -1$" must be included in the first clause of the definition as in this case there simply is no 'γ'; this is in accordance with the θ-component of E_0 being the K-vector space K_θ . The only thing that remains in order to see that E_0 is an R^α-module is to verify the associativity of the operation just defined.

Lemma 4.5 *The action of* R^α *on* E_0 *is associative, and hence makes* E_0 *into a left* R^α*-module.*

Proof: Take standard monomials $\eta_0 = \hat{\eta}_0 x_\delta^l$ and $\eta_1 = \hat{\eta}_1 x_\gamma^m$, and inverse standard monomial $a\theta^{-1} = aX_\beta^{-n}\hat{\theta}^{-1}$. We need to verify that $\eta_0 \centerdot (\eta_1 \centerdot a\theta^{-1}) = (\eta_0\eta_1) \centerdot a\theta^{-1}$. We have to consider a variety of cases depending on the ordering of $\{\beta, \gamma, \delta\}$, and, where appropriate, the comparisons among the exponents l, m, and n. In any case where δ or γ is greater than β, or one of them equals β and the corresponding exponent is greater than n, both sides of the associative law are eaily seen to reduce to 0. I leave it to the reader to verify that the various degenerate subcases arising from the order of η_0 or of η_1 being -1, or possibilities like $m-n=0$ in case (2), are either trivial or handled correctly by the following computations. So we assume as induction hypothesis that

$$\eta'_0 \cdot (\eta'_1 \cdot a\theta^{-1}) = (\eta'_0\eta'_1) \cdot a\theta^{-1}$$

for all $a\theta^{-1}$, whenever $\mathrm{ord}(\eta'_0) \leq \mathrm{ord}(\eta_0)$ and $\mathrm{ord}(\eta'_1) \leq \mathrm{ord}(\eta_1)$, with at least one of the inequalities being strict.

Case 1 $(\delta, \gamma < \beta)$

$$\begin{aligned}
\eta_0 \cdot (\eta_1 \cdot a\theta^{-1}) &= \eta_0 \cdot \rho_\theta(\eta_1)a\theta^{-1} && (\text{since } \gamma < \beta) \\
&= \rho_\theta(\eta_0)\rho_\theta(\eta_1)a\theta^{-1} && (\text{since } \delta < \beta) \\
&= \rho_\theta(\eta_0\eta_1)a\theta^{-1} \\
&= (\eta_0\eta_1) \cdot a\theta^{-1}
\end{aligned}$$

the latter since the order of the standard form of $\eta_0\eta_1$ is less than β by Lemma 1.1.

Case 2 $(\delta < \gamma = \beta)$

$$\begin{aligned}
\eta_0 \cdot (\eta_1 \cdot a\theta^{-1}) &= \eta_0 \cdot (\hat{\eta}_1 \cdot aX_\beta^{m-n}\hat{\theta}^{-1}) && (\text{clause 2 of the definition}) \\
&= (\eta_0\hat{\eta}_1) \cdot aX_\beta^{m-n}\hat{\theta}^{-1} && (\text{induction hypothesis})
\end{aligned}$$

$$\begin{aligned}
(\eta_0\eta_1) \cdot a\theta^{-1} &= (\eta_0\hat{\eta}_1 x_\beta^m) \cdot a\theta^{-1} \\
&= (\eta_0\hat{\eta}_1) \cdot aX_\beta^{m-n}\hat{\theta}^{-1} && (\text{clause 2 of the definition})
\end{aligned}$$

the latter since the order of the standard form of $\eta_0\hat{\eta}_1$ is less than β.

Case 3 $(\gamma < \delta = \beta)$

$$\begin{aligned}
\eta_0 \cdot (\eta_1 \cdot a\theta^{-1}) &= \eta_0 \cdot \rho_\theta(\eta_1)a\theta^{-1} && (\text{since } \gamma < \beta) \\
&= \hat{\eta}_0 \cdot \rho_\theta(\eta_1)aX_\beta^{l-n}\hat{\theta}^{-1} && (\text{clause 2 of the definition})
\end{aligned}$$

$$\begin{aligned}
(\eta_0\eta_1) \cdot a\theta^{-1} &= (\hat{\eta}_0 x_\beta^l \eta_1) \cdot a\theta^{-1} \\
&= (\hat{\eta}_0 \rho_\beta^l(\eta_1) x_\beta^l) \cdot a\theta^{-1} && (\text{since } \gamma < \beta) \\
&= \hat{\eta}_0 \cdot (\rho_\beta^l(\eta_1) x_\beta^l \cdot a\theta^{-1}) && (\text{induction hypothesis}) \\
&= \hat{\eta}_0 \cdot (\rho_\beta^l(\eta_1) \cdot aX_\beta^{l-n}\hat{\theta}^{-1}) && (\text{clause 2 of the definition}) \\
&= \hat{\eta}_0 \cdot \rho_{\hat{\theta}}\Big(\rho_\beta^{n-l}(\rho_\beta^l(\eta_1))\Big) aX_\beta^{l-n}\hat{\theta}^{-1} && (\text{clause 1 of the definition}) \\
&= \hat{\eta}_0 \cdot \rho_\theta(\eta_1) aX_\beta^{l-n}\hat{\theta}^{-1}
\end{aligned}$$

Case 4 $(\delta = \gamma = \beta)$

$$\begin{aligned}
\eta_0 \centerdot (\eta_1 \centerdot a\theta^{-1}) &= \eta_0 \centerdot (\hat{\eta}_1 \centerdot aX_\beta^{m-n}\hat{\theta}^{-1}) && \text{(clause 2 of the definition)} \\
&= (\eta_0\hat{\eta}_1) \centerdot aX_\beta^{m-n}\hat{\theta}^{-1} && \text{(induction hypothesis)} \\
&= \rho_{\eta_0}(\hat{\eta}_1)\eta_0 \centerdot aX_\beta^{m-n}\hat{\theta}^{-1} && \text{(since } \mathrm{ord}(\hat{\eta}_1) < \mathrm{ord}(\eta_0)\text{)} \\
&= \rho_{\eta_0}(\hat{\eta}_1)\hat{\eta}_0 \centerdot aX_\beta^{l+m-n}\hat{\theta}^{-1} && \text{(clause 2 of the definition)}
\end{aligned}$$

$$\begin{aligned}
(\eta_0\eta_1) \centerdot a\theta^{-1} &= (\eta_0\hat{\eta}_1 x_\beta^m) \centerdot a\theta^{-1} \\
&= (\rho_{\eta_0}(\hat{\eta}_1)\eta_0 x_\beta^m) \centerdot a\theta^{-1} && \text{(since } \mathrm{ord}(\hat{\eta}_1) < \mathrm{ord}(\eta_0)\text{)} \\
&= (\rho_{\eta_0}(\hat{\eta}_1)\hat{\eta}_0 x_\beta^{l+m}) \centerdot a\theta^{-1} \\
&= \rho_{\eta_0}(\hat{\eta}_1)\hat{\eta}_0 \centerdot aX_\beta^{l+m-n}\hat{\theta}^{-1} && \text{(clause 2 of the definition)}
\end{aligned}$$

∎

Now I begin to consider the task of solving linear equations in E_0. In particular I will show that for $s \in S$ and for any $e \in E_0$, the equation $s.v = e$ has a unique solution in E_0. Thus E_0 is in fact an $\mathcal{R}^\alpha$-module. I will also develop the tools for finding all solutions to division problems.

Lemma 4.6 *Scalar multiplication cannot increase the degree. More precisely, let η, $\theta \in \Theta$. Then*

1. $\deg(\eta \centerdot \theta^{-1}) \leq \deg(\theta^{-1})$.
2. *If* $\mathrm{ord}(\eta) = -1$ *or* $\mathrm{ord}(\eta) < \mathrm{ord}(\theta)$ *then* $\deg(\eta \centerdot \theta^{-1}) = \deg(\theta^{-1})$.
3. *If* $\mathrm{ord}(\eta) \geq 0$ *and* $\mathrm{ord}(\eta) \geq \mathrm{ord}(\theta)$ *then* $\deg(\eta \centerdot \theta^{-1}) < \deg(\theta^{-1})$.
4. *If* $\deg(\eta) > \deg(\theta)$, *then* $\eta \centerdot \theta^{-1} = 0$.

Proof: Immediate by the definition of multiplication. ∎

Lemma 4.7 *For any $s \in S$ the unique solution to $s.v = 0$ in E_0 is $v = 0$.*

Proof: Since s has a non-zero constant term s_0, it follows from Lemma 4.6 that if e_0 is the term of e of largest degree, then $s_0 \centerdot e_0$ is the term of $s \centerdot e$ of largest degree; that is, if $e \neq 0$ then $s \centerdot e \neq 0$. ∎

Lemma 4.8 *If $0 \neq e$ with $a_0\theta_0^{-1}$ being the term of e of largest degree, then* $\mathrm{ann}(e) = \langle\langle x_0\theta_0 \rangle\rangle$.

Proof: If $a\theta^{-1}$ is any other term of e then already $\theta_0 \centerdot a\theta^{-1} = 0$ by Lemma 4.6. Clearly $x_0\theta_0 \centerdot a\theta_0^{-1} = 0$. On the other hand, if $r \centerdot e = 0$ we can write r in the form $s\theta_1$ with $s \in S$ and $\theta_1 \in \Theta$. Then $s\theta_1 \centerdot e = 0$ implies that $\theta_1 \centerdot e = 0$ by Lemma 4.7. Clearly if $\theta_1 \preceq \theta_0$ then $\theta_1 \centerdot a\theta_0^{-1} \neq 0$ by the definition of multiplication; and so by Lemma 4.6 $\theta_1 \centerdot e \neq 0$. Thus $\theta_1 \succ \theta_0$, that is, $\theta_1 \in \langle\langle x_0\theta_0 \rangle\rangle$. ∎

Lemma 4.9 *For $\eta \in \Theta$, the solutions in E_0 to the equation $\eta \centerdot v = 0$ are exactly the elements of $\sum_{\theta<\eta} K_\theta \theta^{-1}$, that is, the elements of lesser degree than η.*

Proof: Obvious. ∎

Lemma 4.10 *To solve all equations of the form $r \centerdot v = e$ ($r \in R^\alpha$, $e \in E_0$) in E_0 it suffices to be able to solve all equations of the forms $s \centerdot v = a\theta^{-1}$ and $\eta \centerdot v = a\theta^{-1}$, where $s \in S$, $\theta \in \Theta$, $\eta \in \Theta$, and the situations $a = 0$, $\theta = 1$ are allowed.*

Proof: If $e \neq 0$, express e as a sum of terms $e = \sum_{i<n} a_i\theta_i^{-1}$. Clearly if we have solutions to each equation $r \centerdot v = a_i\theta_i^{-1}$ then we get a solution to the original equation as a sum of these. Any $r \in R^\alpha$ can be written in the form $r = s\eta$ where $s \in S$ and $\eta \in \Theta$. Thus $r \centerdot v = e$ if and only if $s \centerdot (\eta \centerdot v) = e$. So if we can solve $s \centerdot w = e$ and $\eta \centerdot v = w$, then we can solve $r \centerdot v = e$. Combining the two reductions yields the Lemma. ∎

Lemma 4.11 *Every equation of the form $\eta \centerdot v = a\theta^{-1}$, $\eta \in \Theta$, has a solution in E_0; in particular a canonical solution (when $a \neq 0$) is found as follows: Write $a\theta^{-1} = aX_\gamma^{-n}\hat{\theta}^{-1}$ and split up the standard form of η as $\eta = \eta_0 x_\gamma^m \eta_1$ (with any of $\eta_0 = 1$, $m = 0$ and $\eta_1 = 1$ allowed). Then $v = [\rho_\theta(\eta_0)]^{-1} a\eta_1^{-1} X_\gamma^{-n-m}\hat{\theta}^{-1}$ is a solution.*

Proof: Since $a\theta^{-1}$ is a solution to $\theta \centerdot v = a$, $\eta \centerdot v = a\theta^{-1}$ implies that $(\theta\eta) \centerdot v = a$. Simple computations from the definition of multiplication give the standard form of $(\theta\eta)^{-1}$ as above. ∎

Note that the formula given simplifies considerably if $\mathrm{ord}(\eta) < \mathrm{ord}(\theta)$: a solution is $v = [\rho_\theta(\eta)]^{-1} a\theta^{-1}$. We get all solutions to $\eta \centerdot v = a\theta^{-1}$ by combining the above with Lemma 4.9. Note that the degree of any solution is the degree of $\theta\eta$.

Lemma 4.12 *Every equation of the form* $s \centerdot v = a\theta^{-1}$, $s \in S$, *has a solution in* E_0, *and the degree of the solution is* $\deg(\theta)$.

Proof: I prove this by induction on $\deg(\theta)$. If $\deg(\theta) = 0$ then $a\theta^{-1} = a \in K$; write $s = s_0 + s_1$ where $s_0 \in K$ and $s_1 \notin S$; then $s.v = a$ has the solution $v = s_0^{-1}a$.

So assume that $\deg(\theta) \geq 1$ (so that $\text{ord}(\theta) \geq 0$) and that $s \centerdot v = a\theta_0^{-1}$ has a solution in E_0 for all $s \in S$ and all $a\theta_0$, $\theta_0 \in \Theta$, of degree less than $\deg(\theta)$. Write $s = s_0 + s_1$ where s_0 is the sum of all terms of s of order less than $\text{ord}(\theta)$; hence every term of s_1 has order $\geq \text{ord}(\theta)$ or $s_1 = 0$. Clearly $s_0 \in S$. Let $c = [\rho_\theta(s_0)]^{-1}a$. Let $e = a\theta^{-1} - s \centerdot c\theta^{-1}$. Then $e = a\theta^{-1} - s_0 \centerdot c\theta^{-1} - s_1 \centerdot c\theta^{-1} = a\theta^{-1} - \rho_\theta(s_0)c\theta^{-1} - s_1 \centerdot c\theta^{-1}$ (since the order of each term of s_0 is less than the order of θ^{-1}), and this is then equal to $a\theta^{-1} - a\theta^{-1} - s_1 \centerdot c\theta^{-1} = -s_1 \centerdot c\theta^{-1}$. Notice that if $s_1 = 0$ then we are done. Otherwise, the order of each term of s_1 is greater than the order of θ_0 so by Lemma 4.6 the degree of each term of $e = -s_1 \centerdot c\theta^{-1}$ is less than the degree of θ. Hence by Lemma 4.10 and the induction hypothesis, the equation $s \centerdot w = e$ has a solution e_0 in E_0, of degree less than $\deg(\theta)$. Then $s \centerdot (c\theta^{-1} + e_0) = s \centerdot c\theta^{-1} + s \centerdot e_0 = s \centerdot c\theta^{-1} + e = s \centerdot c\theta^{-1} + a\theta^{-1} - s \centerdot c\theta^{-1} = a\theta^{-1}$. Clearly $\deg(c\theta^{-1} + e_0) = \deg(\theta)$. ∎

Note that the proofs of the preceding lemmas actually give an algorithm for solving any equation $r \centerdot v = e$ in the R^α-module E_0. However the obvious restrictions on the allowable degrees of terms in the answer give an easier method of computing solutions. Write down a formal sum with undetermined coefficients of the finitely many monomials that could actually appear in a solution and are not annihilated by r; multiply formally by r; solve by comparing coefficients to the representation of e as a inverse polynomial. Unfortunately this more efficient method of solving equations does not yield an efficient proof of the desired result following. For to use this method to prove the main theorem, we would be need to be able to show that the systems of linear equations over K arising by comparing coefficients are always consistent.

Corollary 4.13 E_0 *is a divisible* $\mathcal{R}^\alpha$*-module.* ∎

Now it is trivial to see that E_0 is an essential extension, and hence the injective envelope, of $\mathcal{R}^\alpha/J$.

Lemma 4.14 E_0 *is an essential extension of* $\mathcal{R}^\alpha/J$.

Proof: Let $0 \neq e \in E_0$ and let $k_0\theta_0^{-1}$ be the term of e of highest degree. Then if $k\theta^{-1}$ is any other term of e, $\theta_0 \centerdot k\theta^{-1} = 0$. Hence $\theta_0 \centerdot e = \theta_0 \centerdot k_0\theta_0^{-1} = k_0$, a non-zero element of $\mathcal{R}^\alpha/J$. ∎

Theorem 4.15 *E_0 is the injective envelope of $\mathcal{R}^\alpha/J$.* ∎

Exercise 4.16 *Find all solutions to the following system of linear equations over* $\mathrm{E}(\mathcal{R}^\alpha/J)$.

$$\begin{array}{rcl} x_0 \centerdot u + x_1 \centerdot v &=& X_1^{-1} \\ u + (x_0+x_1) \centerdot v &=& 1 + X_1^{-1} \end{array}$$

4.2 All the other indecomposable injectives

Now the descriptions of all the $\mathrm{E}(\mathcal{R}^\alpha/\mathcal{R}^\alpha x_\beta)$, $0 < \beta < \alpha$, follow in a similar manner. The reader should note the similarities to the commutative ring constructions in [6], in particular the appearance in the descriptions of the quotient fields K^β, taken over successively larger subdomains of $\mathcal{R}^\alpha$. This is an indication that in spite of the many peculiarities exhibited by both the ring and its indecomposable injectives, the $\mathrm{E}(\mathcal{R}^\alpha/\mathcal{R}^\alpha x_\beta)$ are in many ways quite well behaved injective modules.

First note that since $\mathcal{R}^\alpha$ is an Ore domain, it has a quotient skew field K^α, and this quotient field is, as a left $\mathcal{R}^\alpha$-module, the injective envelope of $\mathcal{R}^\alpha$.

Theorem 4.17 *The injective envelope of $\mathcal{R}^\alpha$ is the quotient skew field of $\mathcal{R}^\alpha$.*

Proof: K^α is clearly an essential extension of $\mathcal{R}^\alpha$, and as already noted, it suffices to check divisibility. But this is obvious. ∎

So for the moment, I let β be any ordinal, $0 < \beta < \alpha$. I begin, as in the previous section, by considering the effects of monomial scalar multiplication on the solutions to monomial equations.

Note that since K^β is clearly an essential extension of its $\mathcal{R}^\alpha$-submodule $\mathcal{R}^\alpha/\mathcal{R}^\alpha x_\beta$, we might as well asssume that the constants of these equations are in K^β. So consider an equation $\theta \centerdot v = a$ with $\theta \in \Theta$ and $a \in K^\beta$. Split the standard monomial θ into two (possibly empty) parts, $\theta = \theta_0\theta_1$ with $\theta_0 \in \bar{R}_\beta$ and $\theta_1 \in R^\alpha x_\beta$. Then any solution to $\theta \centerdot v = a$ is also a solution to $\theta_1 \centerdot v = (1/\theta_0)a$ and conversely. So we might as well assume that $\theta \succeq x_\beta$.

(The reader should be careful in the sequel about the distinction between reciprocals in K^β such as $(1/\theta_0)$ above, and the formal inverse monomials that will be introduced later; in this context θ_1^{-1} will be one such.)

Lemma 4.18 *Let θ and η be standard monomials, $\theta \succeq x_\beta$, and let $a \in K^\beta$. Let e be any solution in $\mathrm{E}(\mathcal{R}^\alpha/\mathcal{R}^\alpha x_\beta)$ to $\theta \centerdot v = a$. Then:*

1. *$\eta \centerdot a = \eta a$ if $\eta \prec x_\beta$ and $\eta \centerdot a = 0$ if $\eta \succeq x_\beta$.*
2. *If $\mathrm{ord}(\eta) < \mathrm{ord}(\theta)$ then $\eta \centerdot e$ is a solution to $\theta \centerdot v = \rho_\theta(\eta)a$.*
3. *If $\mathrm{ord}(\eta) = \mathrm{ord}(\theta)$, (and so $\eta \succeq x_\beta$), say $\eta = \hat{\eta}x_\gamma^m$ and $\theta = \hat{\theta}x_\gamma^n$, and $m \leq n$ then $\eta \centerdot e = \hat{\eta} \centerdot \hat{e}$, where $\hat{e}$ is a solution to $\hat{\theta}x_\gamma^{n-m} \centerdot v = a$.*
4. *if $\mathrm{ord}(\eta) = \mathrm{ord}(\theta)$ as in (iii) and $m > n$ or if $\mathrm{ord}(\eta) > \mathrm{ord}(\theta)$, (and so $\mathrm{ord}(\eta) > \mathrm{ord}(x_\beta)$), then $\eta \centerdot e = 0$.*

Proof: The same as the proof of lemma 4.1. ∎

Definition 4.19 *Let $\Theta_\beta = \{\, \theta \in \Theta \colon$ no x_γ, $\gamma < \beta$ occurs in $\theta \,\}$ (and so $1 \in \Theta_\beta$). For $\theta \in \Theta_\beta$, let K_θ^β be the K^β-vector space on K^β with scalar action $t^{-1}r \centerdot k = [\rho_\theta(t)]^{-1}\rho_\theta(r)k$, for $0 \neq t, r \in \bar{R}_\beta$.*

Let $E_\beta = \bigoplus_{\theta\in\Theta_\beta} K_\theta^\beta \theta^{-1}$.

Define an action of R^α on E_β as before:

$$\eta \centerdot a\theta^{-1} = \begin{cases} \rho_\theta(\eta)a\theta^{-1} & \mathrm{ord}(\eta) < \mathrm{ord}(\theta)\,, \\ \hat{\eta} \centerdot aX_\beta^{m-n}\hat{\theta}^{-1} & \mathrm{ord}(\eta) = \mathrm{ord}(\theta)\,, \eta = \hat{\eta}x_\gamma^m \text{ and } \theta = \hat{\theta}x_\gamma^n\,, m \leq n\,, \\ 0 & \mathrm{ord}(\eta) = \mathrm{ord}(\theta)\,, m > n\,, \text{ or } \mathrm{ord}(\eta) > \mathrm{ord}(\theta)\,. \end{cases}$$

Lemma 4.20 1. *E_β is a left R^α-module.*

2. *E_β is a divisible left $\mathcal{R}^\alpha$-module.*
3. *E_β is an essential extension of K^β.*

Proof: These follow for exactly the same reasons as before. . ∎

Theorem 4.21 *E_β is the injective envelope of $\mathcal{R}^\alpha/\mathcal{R}^\alpha x_\beta$.* ∎

Note that if I had allowed $\beta = \alpha$ above, we would have $\Theta_\beta = \{\,1\,\}$, so naming $\mathrm{E}(\mathcal{R}^\alpha)$ as E_α is consistent with Definition 4.19.

4.3 Applications

For the most part, I leave it to the reader to explore possible applications of these descriptions. Computational problems such as solving systems of linear equations, while daunting, are feasible. Extracting more useful algebraic (or model-theoretic) information is of course more of a challenge. In these examples once again, the (algebraic) socle series stabilizes after only ω steps, whereas the elementary socle series quite naturally corresponds to the structure of the description.

Proposition 4.22 *(socle series)*

1. $\mathrm{soc}_n(E_0) = \sum_{m<n} K_{x_0^m} X_0^{-m}$.
2. *For all* $\gamma \geq \omega$, $\mathrm{soc}_\gamma(E_0) = \sum_{m<\omega} K_{x_0^m} X_0^{-m}$.

Proof: The key point is that $\rho_0 : K \to K$ is an embedding of a field into K , whereas for $\beta > 0$, $\rho_\beta : \bar{R}_\beta \to K$ is an embedding of a domain not a field into K . So, for instance, the minimal submodules over $\mathrm{soc}_1(E_0) = K$ are of the form $K + \rho_0[K]aX_0^{-1}$ for some $a \in K$. For $b \in \rho_0[K]a$ if and only if $b = da$ for some $d \in \rho_0[K]$; and since $\rho_0[K]$ is a subfield, $d^{-1} \in \rho_0[K]$ so $d^{-1}b = a$ and $a \in \rho_0[K]b$. On the other hand, $(\rho_1(x_0))^{-1}$ is not in the image of ρ_1 and so $\langle\langle X_1^{-1}\rangle\rangle$, $\langle\langle \rho_1(x_0)X_1^{-1}\rangle\rangle$, $\langle\langle \rho_1(x_0^2)X_1^{-1}\rangle\rangle$,... form an infinite descending chain of submodules with intersection K , but no one of them is contained in $\mathrm{soc}_\omega(E_0)$. ∎

Proposition 4.23 *(elementary socle series)* $\mathrm{soc}^\delta(E_0) = \sum_{\theta, \deg(\theta)<\delta} K_\theta \theta^{-1}$ *for* $\delta \leq \omega^\alpha$.

In particular, for $\gamma < \alpha$, $\mathrm{soc}^{\omega^\gamma}(E_0) = \sum_{\theta, \theta \prec x_\gamma} K_\theta \theta^{-1}$.

Proof: The definable subgroups are determined by Lemma 4.9. ∎

I will investigate only one application of substance. We saw in Section 2 that the "uncomplicated" indecomposable ($\mathrm{E}(R/J^\omega)$) was a homomorphic image of the "complicated" one ($\mathrm{E}(R/J)$). This "layering" of the indecomposables is one of the interesting and peculiar features of the example. This sort of structure is displayed beautifully in the general case.

Theorem 4.24 *Let* $0 < \beta < \alpha$ *. Then for each* γ *,* $\beta \leq \gamma < \alpha$ *,* $E_0/\mathrm{soc}^{\omega^\beta}(E_0)$ *has an infinite direct sum of copies of* E_γ *as a direct summand.*

Proof: First note that $A_\beta = \sum_{\theta,\theta\succeq x_\beta} K_\theta \theta^{-1}$ is a set of representatives of the cosets of $\mathrm{soc}^{\omega^\beta}(E_0)$ in E_0. The scalar multiplication of $\mathcal{R}^\alpha$ on A_β is then just ordinary scalar multiplication on E_0, followed by setting to 0 any term of degree less than $\deg(x_\beta)$. Since $\mathcal{R}^\alpha$ is hereditary, $E_0/\mathrm{soc}^{\omega^\beta}(E_0) \cong A_\beta$ is injective. So it suffices to find $\mathcal{R}^\alpha$-linearly independent elements of A_β with annihilator $\mathcal{R}^\alpha x_\gamma$ for any γ, $\beta \leq \gamma < \alpha$.

In fact, for fixed γ, the set $\mathcal{B}{X_\gamma}^{-1}$ will do, where $\mathcal{B}$ is a basis for K_{x_γ} as a left K vector space. For in E_0, $r \,\centerdot\, a{X_\gamma}^{-1}$ either has order that of x_γ, or order -1. So in A_β, $r \,\centerdot\, a{X_\gamma}^{-1} = 0$ if and only if $r \in \mathcal{R}^\alpha x_\gamma$.

Furthermore, if $r \,\centerdot\, a{X_\gamma}^{-1} = 0 \neq 0$ in A_β, the product is of the form $a'{X_\gamma}^{-1}$ in A_β.

So no non-trivial linear combination of such things can be zero. For the same reason, the elements chosen for different γ's are independent from each other. ∎

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