Math 2720 Multivariable Calculus, Term Exam 2, November 9, 2011

Examiner: A. Prymak, exam duration: 50 minutes, total value of all questions: 50 points.

No calculators or other aids permitted.

Student ID:____________________ **Name:** ______________________________

1 [12 pts]. Let $f(x,y) = \tan(-xy) + \log_2(x^2)e^{5y} + e^{\pi}$.

(a) Find the directional derivative of f at the point $(2,0)$ in the direction of the point $(1,-1)$.

$$\frac{\partial f}{\partial x} = \sec^2(-xy)(-y) + 2\frac{1}{x\ln 2}e^{5y}$$

$$\frac{\partial f}{\partial y} = \sec^2(-xy)(-x) + \log_2(x^2)\,5e^{5y}$$

$$\nabla f(2,0) = \left\langle \frac{1}{\ln 2}, -2+10 \right\rangle = \left\langle \frac{1}{\ln 2}, 8 \right\rangle$$

$$\vec{v} = \langle 1,-1\rangle - \langle 2,0\rangle = \langle -1,-1\rangle \qquad \|\vec{v}\| = \sqrt{2}$$

$$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{1}{\sqrt{2}}\langle -1,-1\rangle$$

$$D_{\vec{u}} f(2,0) = \nabla f(2,0)\cdot\vec{u} = \left\langle \frac{1}{\ln 2}, 8\right\rangle \cdot \frac{1}{\sqrt{2}}\langle -1,-1\rangle$$

$$= \underline{-\frac{1}{\sqrt{2}}\left(\frac{1}{\ln 2} + 8\right)}$$

(b) What is the largest possible value of a directional derivative of f at $(2,0)$?

$$\|\nabla f(2,0)\| = \underline{\sqrt{\left(\frac{1}{\ln 2}\right)^2 + 64}}$$

2 [9 pts]. Let $f = f(x,y)$ be differentiable function satisfying

$$(x-y)\frac{\partial f}{\partial x} + (x+y)\frac{\partial f}{\partial y} = 0.$$

Prove that after the change of variables $x = r\cos\theta$, $y = r\sin\theta$, this function satisfies

$$r\frac{\partial f}{\partial r} + \frac{\partial f}{\partial \theta} = 0.$$

$$r\frac{\partial f}{\partial r} + \frac{\partial f}{\partial \theta} = r\left(\frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial r}\right) + \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial \theta}$$

$$= r\left(\frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta\right) + \frac{\partial f}{\partial x}r(-\sin\theta) + \frac{\partial f}{\partial y}r\cos\theta$$

$$= \frac{\partial f}{\partial x}\cdot x + \frac{\partial f}{\partial y}\cdot y + \frac{\partial f}{\partial x}(-y) + \frac{\partial f}{\partial y}x$$

$$= \frac{\partial f}{\partial x}(x-y) + \frac{\partial f}{\partial y}(x+y) = 0.$$

3 [4 pts]. Does there exist a function $f(x,y)$ with partial derivatives $f_x(x,y) = 2xy + 4y^2$ and $f_y(x,y) = x^2 - 8yx$? Justify your answer.

<u>No:</u> $\left.\begin{array}{l} f_{xy} = 2x + 8y \\ f_{yx} = 2x - 8y \end{array}\right\}$ - different

but have to be equal by Clairaut's Thm since both are continuous everywhere.

4 [11 pts]. Evaluate each of the following limits or explain why it does not exist.

(a) $\lim_{(x,y)\to(1,1)} \dfrac{x^2 - y^2x + xy - y^3}{\sqrt{x+1} - \sqrt{y^2+1}}$

$$= \lim_{(x,y)\to(1,1)} \frac{x(x-y^2) + y(x-y^2)}{(x+1)-(y^2+1)} \cdot \left(\sqrt{x+1} + \sqrt{y^2+1}\right)$$

$$\overset{\text{cancel } x-y^2}{=} \lim_{(x,y)\to(1,1)} (x+y)\left(\sqrt{x+1} + \sqrt{y^2+1}\right) = \underline{4\sqrt{2}}$$

(b) $\lim_{(x,y)\to(0,0)} \dfrac{x^2y}{x^6 + 4y^3}$

along $x=0$: $\lim_{y\to 0} \dfrac{0}{0+4y^3} = 0$

along $x=y$: $\lim_{x\to 0} \dfrac{x^3}{x^6+4x^3} = \lim_{x\to 0} \dfrac{1}{x^3+4} = \dfrac{1}{4}$

different,

So the limit $\underline{\text{D.N.E.}}$

5 [14 pts]. Use Lagrange multipliers to find the absolute minimum and maximum values of $f(x, y) = x^3 + y^3$ subject to the constraint $x^2 + y^2 = 4$.

$$\begin{cases} 3x^2 = \lambda \cdot 2x \\ 3y^2 = \lambda \cdot 2y \\ x^2 + y^2 = 4 \end{cases}$$

If $x \neq 0$ and $y \neq 0$, then $\begin{cases} 3x = 2\lambda \\ 3y = 2\lambda \end{cases}$, so $x = y$

$x^2 + y^2 = 4$ becomes $2x^2 = 4$ $\quad x = \pm\sqrt{2}$

So, $x = y = \pm\sqrt{2}$ are solutions.

If $x = 0$ then $y^2 = 4$, so $y = \pm 2$. $\quad 3 \cdot 4 = \lambda \cdot 2 \cdot (\pm 2)$

$\lambda = \pm 3$.

Similarly, if $y = 0$, then $x = \pm 2$.

$f(\sqrt{2}, \sqrt{2}) = 2\sqrt{2} + 2\sqrt{2} = 4\sqrt{2}$

$f(-\sqrt{2}, -\sqrt{2}) = -4\sqrt{2}$

$f(0, 2) = 8 \qquad f(2, 0) = 8$

$f(0, -2) = -8 \qquad f(-2, 0) = -8$

Ans: max value is 8, min value is -8.

6 [BONUS, 5 pts]. (Only significant progress will be marked.) Find all values of positive integers m, n such that the limit $\lim_{(x,y)\to(0,0)} \frac{x^m y^n}{x^2+y^2}$ does not exist.

Ans: $m = n = 1$.

First show that it D.N.E. for these m, n:

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} \qquad \left.\begin{array}{l} \text{along } x=0: \lim_{y\to 0} \frac{0\cdot y}{0^2+y^2} = 0 \\ \text{along } x=y: \lim_{x\to 0} \frac{x^2}{x^2+x^2} = \frac{1}{2} \end{array}\right\} \text{diff.}$$

Now show that if $m \geq 2$ or $n \geq 2$ then the lim. exists. Without loss of generality, let $m \geq 2$. Then for $|x| \leq 1$:

$$0 \leq \left| \frac{x^m y^n}{x^2+y^2} \right| = |x|^{m-2} \cdot \frac{x^2}{x^2+y^2} \cdot |y|^n$$

$$\leq \frac{x^2}{x^2+y^2} \cdot |y|^n$$

$$\leq |y|^n \to 0 \text{ as } y \to 0.$$

So, by Squeeze Thm, the limit is zero.