22. Base $n=1$: $x^n - y^n = x - y$ – divisible by $x-y$ – true

Assume that for $n=k$ we have proved that

$$x-y \quad \text{divides} \quad x^k - y^k.$$

We need to prove that $(n=k+1)$

$$x-y \quad \text{divides} \quad x^{k+1} - y^{k+1}.$$

Indeed,
$$\begin{aligned} x^{k+1} - y^{k+1} &= x \cdot x^k - y \cdot y^k \\ &= x(x^k - y^k) + xy^k - y \cdot y^k \\ &= x\underbrace{(x^k - y^k)}_{\substack{\text{divisible by} \\ x-y \text{ by assumption}}} + \underbrace{(x-y)\,y^k}_{\text{divisible by } x-y.} \end{aligned}$$

So, $x^{k+1} - y^{k+1}$ is divisible by $x-y$, step is completed.

By PMI, we proved the statement for all $n \geq 1$.

23. Base $n=1$: $5^{2+2} - 24 - 25 = 625 - 24 - 25 = 576$ – divisibly by 576.

Assume that 576 divides $5^{2k+2} - 24k - 25$. We need to prove that 576 divides $5^{2k+4} - 24(k+1) - 25$.

$$= 25 \cdot 5^{2k+2} - 24k - 49 = 25(5^{2k+2} - 24k - 25) + 25 \cdot 24k$$
$$+ 25 \cdot 25 - 24k - 49 = 25\underbrace{(5^{2k+2} - 24k - 25)}_{\substack{\text{divisible by } 576 \\ \text{by assumption}}} + \underbrace{576k + 576}_{\substack{\text{clearly divisible} \\ \text{by } 576.}}$$

This completes the step of induction. By PMI, we proved the statement for all $n \geq 1$.

By Ex. 1,3, if $r \neq 1$, then

$$a + ar + ar^2 + \ldots + ar^{k-1} = \frac{a(r^k - 1)}{r-1}.$$

e choose a to be the first term, r – the
uotient of the terms, k so that ar^{k-1} is
he last term of summation.

For #2: $a = 3$, $r = 3$, $k = n$.

For # 8: $a = 2$, $r = 2^2 = 4$, $k = n$.

For #11: $a = \frac{1}{5^2} = \frac{1}{25}$, $r = \frac{1}{5^2} = \frac{1}{25}$, $k = n$.

For #12: $a = \frac{1}{2}$, $r = \frac{1}{2}$, $k = n$.

25. Base $n=0$: $x^{2n+1} + y^{2n+1} = x + y$ – divisible by $x+y$.

Assume that $x+y$ divides $x^{2k+1} + y^{2k+1}$, $k \geq 0$.

We need to prove that $x+y$ divides $x^{2k+3} + y^{2k+3}$.

Indeed, $x^{2k+3} + y^{2k+3} = x^2 \cdot x^{2k+1} + y^2 \cdot y^{2k+1}$

$$= x^2(x^{2k+1} + y^{2k+1}) - x^2 y^{2k+1} + y^2 \cdot y^{2k+1}$$

$$= \underbrace{x^2(x^{2k+1} + y^{2k+1})}_{\text{divisible by } x+y \text{ by assumption}} - \underbrace{(x-y)(x+y)\, y^{2k+1}}_{\text{clearly divisible by } x+y}.$$

So, $x+y$ divides $x^{2k+3} + y^{2k+3}$. By PMI,
this completes the proof.

26. Clearly, n^3+6n^2+2n is a positive integer when n is a positive integer, so we only need to prove that 3 divides n^3+6n^2+2n for all $n \geq 1$.

Base: $n=1$: $1^3+6\cdot 1^2+2\cdot 1=9$ - divisible by 3.

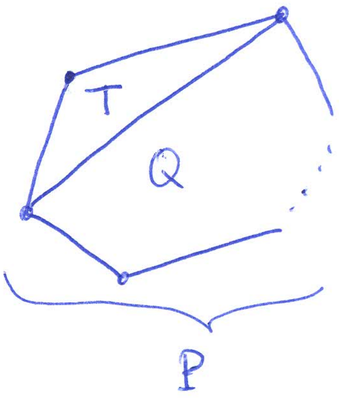
Assume that 3 divides k^3+6k^2+2k. We need to prove that 3 divides $(k+1)^3+6(k+1)^2+2(k+1)$

$$= k^3+3k^2+3k+1 + 6k^2+12k+6+2k+2$$

$$= k^3+9k^2+17k+3 = \underbrace{(k^3+6k^2+2k)}_{\text{divisible by 3 by assumption}} + \underbrace{3k^2+15k+3}_{\text{clearly divisible by 3 - all coefficients are divisible}}$$

So, the step is completed, and by PMI the statement is proved for all $n \geq 1$.

27. Any polygon has at least 3 sides. For $n=3$, any polygon with 3 sides is a triangle, and we know that the sum of the interior angles of any triangle is $\pi = (3-2)\pi = (n-2)\pi$. This verifies the base of induction ($n=3$). Assume that we have proved the statement for ~~any~~ some $n=k \geq 3$. We need to prove for $n=k+1$.



Let P be a polygon with $k+1$ sides. We can replace some two consequtive sides by one side to form a ~~k~~ polygon Q with k sides. If T is ~~the~~ triangle formed by the three sides (two removed from P and one new for Q), then the sum of interior angles of P equals to the sum of interior angles of T (we know it is π) ~~and~~ plus the sum of interior angles of Q, which equals $(k-2)\pi$ by assumption. Adding, we have $\pi + (k-2)\pi = (k-1)\pi = ((k+1)-2)\pi$ as required.

28. If $1+2+\ldots+k = \frac{(k+2)(k-1)}{2}$, we need to show that $1+2+\ldots+(k+1) = \frac{(k+3)\cdot k}{2}$. Indeed, by assumption

$$LHS = (1+2+\ldots+k) + (k+1) = \frac{(k+2)(k-1)}{2} + k+1$$

$$= \frac{k^2+2k-k-2+2k+2}{2} = \frac{k^2+3k}{2} = \frac{(k+3)k}{2} = RHS.$$

The result is NOT valid for all $n \geq 1$ because the base $(n=1)$ fails:

$$1 \neq \frac{(1+2)(1-1)}{2} = 0.$$

29. If $1+3+5+..+(2k-1) = k^2+4$, we need to prove that $1+3+5+..+(2k-1)+(2k+1) = (k+1)^2+4$.

Indeed, by assumption:

$LHS = (1+3+..+(2k-1)) + (2k+1) = k^2+4+2k+1 = k^2+2k+5$

$RHS = k^2+2k+1+4 = k^2+2k+5$.

The proposition is not true for all $n \geq 1$, say for $n=1$: $LHS = 1$, $RHS = 1^2+4 = 5$.

30. Base $n=1$:

$LHS = 1$

$RHS = 1^2+1 = 2$

true: $LHS < RHS$.

Assume for $n=k$ we proved

$$1+3+..+(2k-1) < k^2+k \qquad (*)$$

We need to prove for $n=k+1$:

$$1+3+..+(2k-1)+(2k+1) < (k+1)^2+(k+1) \qquad (**).$$

In order to obtain $(**)$, we need to add $(*)$ and the following inequality:

$$(2k+1) \leq (k+1)^2+(k+1) - k^2 - k \qquad (***).$$

In other words, $(***)$ plus $(*)$ gives $(**)$.

So, we need to check that $(***)$ is true.

$RHS = k^2+2k+1+k+1-k^2-k = 2k+2$

$2k+1 \leq 2k+2$ — true. This completes the step and proof.