1. Use Cramer's rule to find the solution of the system:

$$3x + 5y = 1$$
$$7x + 8y = 1$$

Solution:

$$x = \frac{\det\begin{pmatrix}1 & 5\\ 1 & 8\end{pmatrix}}{\det\begin{pmatrix}3 & 5\\ 7 & 8\end{pmatrix}} = \frac{(1\cdot 8 - 5\cdot 1)}{(3\cdot 8 - 5\cdot 7)} = -\frac{3}{11}; \quad y = \frac{\det\begin{pmatrix}3 & 1\\ 7 & 1\end{pmatrix}}{\det\begin{pmatrix}3 & 5\\ 7 & 8\end{pmatrix}} = \frac{(3\cdot 1 - 1\cdot 7)}{(3\cdot 8 - 5\cdot 7)} = \frac{4}{11}$$

2. Use Cramer's rule to find the solution of the system:

$$17x + 7y + 7z = 1$$
$$7x + 17y + 7z = 0$$
$$7x + 7y + 17z = 0$$

Solution: Simplify the determinant of the coefficient matrix by $R_1 \to R_1 + R_2 + R_3$:

$$\det\begin{pmatrix}17 & 7 & 7\\ 7 & 17 & 7\\ 7 & 7 & 17\end{pmatrix} = \det\begin{pmatrix}31 & 31 & 31\\ 7 & 17 & 7\\ 7 & 7 & 17\end{pmatrix} = 31\cdot\det\begin{pmatrix}1 & 1 & 1\\ 7 & 17 & 7\\ 7 & 7 & 17\end{pmatrix} = 31\cdot\det\begin{pmatrix}1 & 1 & 1\\ 0 & 10 & 0\\ 0 & 0 & 10\end{pmatrix} = 3100$$

Then, by Cramer's rule,

$$x = \frac{1}{3100}\cdot\det\begin{pmatrix}1 & 7 & 7\\ 0 & 17 & 7\\ 0 & 7 & 17\end{pmatrix} = \frac{1}{3100}\cdot\det\begin{pmatrix}17 & 7\\ 7 & 17\end{pmatrix} = \frac{240}{3100} = \frac{24}{310},$$

$$y = \frac{1}{3100}\cdot\det\begin{pmatrix}17 & 1 & 7\\ 7 & 0 & 7\\ 7 & 0 & 17\end{pmatrix} = \frac{-1}{3100}\cdot\det\begin{pmatrix}7 & 7\\ 7 & 17\end{pmatrix} = \frac{-70}{3100} = \frac{-7}{310},$$

$$z = \frac{1}{3100}\cdot\det\begin{pmatrix}17 & 7 & 1\\ 7 & 17 & 0\\ 7 & 7 & 0\end{pmatrix} = \frac{1}{3100}\cdot\det\begin{pmatrix}7 & 17\\ 7 & 7\end{pmatrix} = \frac{-70}{3100} = \frac{-7}{310}.$$

3. It is known that, for some missing value of a, the system:

$$ax - z = 1$$
$$3x + y - w = 0$$
$$x + 2z + 2w = 0$$
$$-x + 2y + 5w = 0$$

is inconsistent. Find the missing value of a.

Solution: First find the determinant of the coefficient matrix. One way to simplify the determinant is to generate three zeros in the second column by applying the row operation $R_4 \to R_4 - 2R_2$:

$$\det\begin{pmatrix}a & 0 & -1 & 0\\ 3 & 1 & 0 & -1\\ 1 & 0 & 2 & 2\\ -1 & 2 & 0 & 5\end{pmatrix} = \det\begin{pmatrix}a & 0 & -1 & 0\\ 3 & 1 & 0 & -1\\ 1 & 0 & 2 & 2\\ -7 & 0 & 0 & 7\end{pmatrix} = \det\begin{pmatrix}a & -1 & 0\\ 1 & 2 & 2\\ -7 & 0 & 7\end{pmatrix}$$

$$= a\cdot\det\begin{pmatrix}2 & 2\\ 0 & 7\end{pmatrix} + \det\begin{pmatrix}1 & 2\\ -7 & 7\end{pmatrix} = 14a + 21$$

If $14a + 21 \neq 0$, then the determinant would be a non-zero number. In this case the system could be solved by Cramer's rule, i.e., the system would be consistent. Since the system is inconsistent, we must have $14a + 21 = 0$, or $a = -3/2$.

4. Let

$$\vec{u} = 5\vec{\imath} - 3\vec{\jmath} - 4\vec{k}$$
$$\vec{v} = -2\vec{\imath} + \vec{\jmath} + 2\vec{k}$$
$$\vec{w} = -4\vec{\imath} + \vec{\jmath} + a\vec{k}$$

(a) What value(s) of a make the vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$ linearly dependent?

Solution: Let A be the square matrix whose columns are the components of $\vec{u}$, $\vec{v}$ and $\vec{w}$, respectively. The vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$ are linearly dependent if and only if the determinant of A is zero. This determinant can be evaluated, for example, by generating two zeros in the second column via $R_1 \to R_1 + 2R_2$ and $R_3 \to R_3 - 2R_2$:

$$\det\begin{pmatrix} 5 & -2 & -4 \\ -3 & 1 & 1 \\ -4 & 2 & a \end{pmatrix} = \det\begin{pmatrix} -1 & 0 & -2 \\ -3 & 1 & 1 \\ 2 & 0 & a-2 \end{pmatrix} = \det\begin{pmatrix} -1 & -2 \\ 2 & a-2 \end{pmatrix} = 6 - a\,.$$

Therefore the vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$ are linearly dependent for $a = 6$.

(b) Pick a value for a that makes the vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$ linearly dependent. Then express one of the three vectors as a linear combination of the other two.

Solution: By part (a), we must pick $a = 6$. Now find the scalars C_1, C_2 and C_3 such that $C_1\vec{u} + C_2\vec{v} + C_3\vec{w} = \vec{0}$. This amounts to solving the system

$$\begin{aligned} 5C_1 - 2C_2 - 4C_3 &= 0 \\ -3C_1 + C_2 + C_3 &= 0 \\ -4C_1 + 2C_2 + 6C_4 &= 0\,. \end{aligned}$$

By Gauss-Jordan elimination,

$$\left(\begin{array}{ccc|c} 5 & -2 & -4 & 0 \\ -3 & 1 & 1 & 0 \\ -4 & 2 & 6 & 0 \end{array}\right) \xrightarrow{R_1 \to R_1 + R_3} \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ -3 & 1 & 1 & 0 \\ -4 & 2 & 6 & 0 \end{array}\right) \xrightarrow[R_3 \to R_3 + 4R_1]{R_2 \to R_2 + 3R_1} \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 7 & 0 \\ 0 & 2 & 14 & 0 \end{array}\right)$$

$$\xrightarrow{R_3 \to R_3 - 2R_3} \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 7 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \implies C_1 = -2C_3 \quad \text{and} \quad C_2 = -7C_3\,,$$

where C_3 is a free variable. Pick $C_3 = 1$. Then $C_1 = -2$ and $C_2 = -7$. Therefore $-2\vec{u} - 7\vec{v} + \vec{w} = \vec{0}$, and so $\vec{w} = 2\vec{u} + 7\vec{v}$.

(c) Pick a value for a that makes the vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$ linearly dependent. Give a geometric description of all linear combinations of $\vec{u}$, $\vec{v}$ and $\vec{w}$.

Solution: By part (a), we must pick $a = 6$. Let $\vec{X} = x\vec{\imath} + y\vec{\jmath} + z\vec{k}$ be any linear combination of $\vec{u}$, $\vec{v}$ and $\vec{w}$. Then $\vec{X} = \alpha\vec{u} + \beta\vec{v} + \gamma\vec{w}$ for some scalars α, β, γ. By the result from part (b),

$$\vec{X} = \alpha\vec{u} + \beta\vec{v} + \gamma(2\vec{u} + 7\vec{v}) = (\alpha + 2\gamma)\vec{u} + (\beta + 7\gamma)\vec{v} = s\vec{u} + t\vec{v}\,,$$

where $s = \alpha + 2\gamma$ and $t = \beta + 7\gamma$. Thus any linear combination of $\vec{u}$, $\vec{v}$ and $\vec{w}$ reduces to a linear combination of $\vec{u}$ and $\vec{v}$. Furthermore, any linear combination of $\vec{u}$ and $\vec{v}$ can be written as

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = s\begin{pmatrix} 5 \\ -3 \\ -4 \end{pmatrix} + t\begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}, \quad \text{or,} \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + s\begin{pmatrix} 5 \\ -3 \\ -4 \end{pmatrix} + t\begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix},$$

where s and t are parameters. Since $\vec{u}$ and $\vec{v}$ are not parallel (one is not a scalar multiple of the other), this is a parametric equation in vector form representing a plane in $\mathbb{E}^3$. The plane passes through $(0,0,0)$ and is parallel to $\vec{u}$ and $\vec{v}$.

(d) Pick a value for a that makes the vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$ linearly independent. Give a geometric description of all linear combinations of $\vec{u}$, $\vec{v}$ and $\vec{w}$.

Solution: Let $a \neq 6$. A vector $\vec{X} = x\vec{i} + y\vec{j} + z\vec{k}$ is a linear combination of $\vec{u}$, $\vec{v}$ and $\vec{w}$ if and only if there exist scalars C_1, C_2 and C_3 such that $C_1\vec{u}+C_2\vec{v}+C_3\vec{w} = \vec{X}$. This is equivalent to the system

$$\begin{aligned} 5C_1 - 2C_2 - 4C_3 &= x \\ -3C_1 + C_2 + C_3 &= y \\ -4C_1 + 2C_2 + aC_4 &= z \end{aligned}$$

having a solution. Note that the columns of the coefficient matrix are the vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$, respectively. Since $a \neq 6$, the vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$ are linearly independent, and so the determinant of the coefficient matrix is not zero. By Cramer's rule, the system will have a unique solution, regardless of the chosen values for x, y and z. Therefore any vector $\vec{X}$ in $\mathbb{E}^3$ can be expressed as a linear combination of $\vec{u}$, $\vec{v}$ and $\vec{w}$, or in other words, the linear combinations of $\vec{u}$, $\vec{v}$ and $\vec{w}$ represent the whole space $\mathbb{E}^3$.