1. Let $\mathcal{P}$ be the plane passing through the points $A(1,1,1)$, $B(2,0,0)$ and $C(1,6,3)$. Let ℓ be the line through the point $D(4,5,16)$ and perpendicular to the plane $\mathcal{P}$.

(a) Find an equation of the plane $\mathcal{P}$ in standard form.

Solution: The vectors

$$\overrightarrow{AB} = (2-1)\vec{\imath} + (0-1)\vec{\jmath} + (0-1)\vec{k} = \vec{\imath} - \vec{\jmath} - \vec{k}$$

and

$$\overrightarrow{AC} = (1-1)\vec{\imath} + (6-1)\vec{\jmath} + (3-1)\vec{k} = 5\vec{\jmath} + 2\vec{k}$$

lie in the plane $\mathcal{P}$. Their cross-product is a normal vector for $\mathcal{P}$:

$$\vec{n} = (\vec{\imath} - \vec{\jmath} - \vec{k}) \times (5\vec{\jmath} + 2\vec{k}) = 3\vec{\imath} - 2\vec{\jmath} + 5\vec{k}\,.$$

A point-normal equation of the plane $\mathcal{P}$ can now be found using any of the points in $\mathcal{P}$, say the point $A(1,1,1)$:

$$3(x-1) - 2(y-1) + 5(z-1) = 0\,.$$

By simplifying, we get the equation of the plane $\mathcal{P}$ in standard form:

$$3x - 2y + 5z = 6\,.$$

(b) Find parametric equations of the line ℓ in vector and scalar forms.

Solution: Since ℓ is perpendicular to $\mathcal{P}$, it must be parallel to the normal vector $\vec{n} = 3\vec{\imath} - 2\vec{\jmath} + 5\vec{k}$. Therefore the vector parametric equation of the line ℓ is:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 16 \end{bmatrix} + t \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix}.$$

Hence the scalar parametric equations are:

$$x = 4 + 3t\,,\; y = 5 - 2t\,,\; z = 16 + 5t\,.$$

(c) Find the point of intersection of the plane $\mathcal{P}$ and the line ℓ.

Solution: Since the point of intersection belongs to the line ℓ, it must be of the form $(x,y,z) = (4+3t, 5-2t, 16+5t)$ for some value of the parameter t. Since this point also belongs to the plane $\mathcal{P}$, it must satisfy the equation $3x - 2y + 5z = 6$, that is:

$$3(4+3t) - 2(5-2t) + 5(16+5t) = 6\,.$$

Hence $t = -2$. The point of intersection is $(4+3t, 5-2t, 16+5t) = (-2, 9, 6)$.

(d) Find the distance from the point D to the plane $\mathcal{P}$.

Solution: The distance, d, between the point D and the plane $\mathcal{P}$ is measured along the straight line through D perpendicular to $\mathcal{P}$. This is exactly the distance between $D(4,5,16)$ and the intersection point $(-2,9,6)$ from part (c):

$$d = \sqrt{(4-(-2))^2 + (5-9)^2 + (16-6)^2} = \sqrt{152} = 2\sqrt{38}\,.$$

NOTE: This result can be verified using the formula for the distance between a point (x_0, y_0, z_0) and a plane $Ax + By + Cz = D$: $d = \frac{|Ax_0 + By_0 + Cz_0 - D|}{\sqrt{A^2+B^2+C^2}}$.

2. Find all points of intersection of the planes $\mathcal{P}_1 :\ 7x + 3y - 4z = 2$ and $\mathcal{P}_2 :\ 2x + y - 3z = -3$. Explain the geometrical significance of your answer.

Solution 1: Consider the equations of the two planes as a system of two equations in three unknowns. One way to solve the system is to express y from the second equation:

$$y = -2x + 3z - 3\,,$$

then substitute the result into the first equation:

$$7x + 3(-2x + 3z - 3) - 4z = 2\,,$$

and then solve for x: $x = 11 - 5z$. Hence $y = -2x + 3z - 3 = -2(11 - 5z) + 3z - 3 = 13z - 25$. By setting $z = t$, where t is an arbitrary real number (a parameter), we find the solution of the system in the form:

$$x = 11 - 5t\,,\ y = 13t - 25\,,\ z = t\,.$$

These parametric equations describe a line in $\mathbb{E}^3$.

Solution 2: The normal vectors of $\mathcal{P}_1$ and $\mathcal{P}_2$ are $\vec{n_1} = 7\vec{i}+3\vec{j}-4\vec{k}$ and $\vec{n_2} = 2\vec{i}+\vec{j}-3\vec{k}$, respectively. If the two planes intersect in a line ℓ, then ℓ is perpendicular to both $\vec{n_1}$ and $\vec{n_2}$. Hence the cross-product of the normal vectors

$$\vec{n_1} \times \vec{n_2} = (7\vec{i} + 3\vec{j} - 4\vec{k}) \times (2\vec{i} + \vec{j} - 3\vec{k}) = -5\vec{i} + 13\vec{j} + \vec{k}$$

must be parallel to ℓ. In order to get the equations of the line ℓ, we only need to find one point belonging to ℓ. For example, if we set $z = 0$ in the two equations for $\mathcal{P}_1$ and $\mathcal{P}_2$, solving for x and y results in the point $(x, y, z) = (11, -25, 0)$. Hence the parametric equations of the intersection line are: $x = 11 - 5t$, $y = -25 + 13t$, $z = t$.

NOTE: If the two planes were parallel, the normal vectors $\vec{n_1}$ and $\vec{n_2}$ would be parallel, and then their cross-product would have to be equal to the zero vector. Since $\vec{n_1} \times \vec{n_2} = -5\vec{i} + 13\vec{j} + \vec{k}$ is not the zero vector, this tells us that the two planes are not parallel, so they must intersect in a straight line.

3. Given the matrices

$$A = \begin{pmatrix} 2 & 5 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\quad B = \begin{pmatrix} 1 & 2 & 2 \\ 0 & -1 & 1 \end{pmatrix},\quad C = \begin{pmatrix} 2 & -5 & 1 \end{pmatrix},\quad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$E = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 2 & 0 \end{pmatrix},\quad F = \begin{pmatrix} 3 & 4 \\ 0 & -1 \\ 0 & 0 \end{pmatrix},\quad G = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},\quad \text{and} \quad H = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -4 \end{pmatrix},$$

(a) identify the matrices of each of the following types:
square, diagonal, identity, zero, column, row, upper triangular, lower triangular;
Solution: square matrices: A, D, E, H ; diagonal matrices: D and H ; identity matrix: D ; zero matrix: G ; column matrix: G ; row matrix: C ; upper triangular matrices: A, D, H ; lower triangular matrices: D, E, H.

(b) evaluate or declare as undefined: $B^T - F$, $C + G$, $3E + 2H$.
Solution:

$$B^T - F = \begin{pmatrix} -2 & -4 \\ 2 & 0 \\ 2 & 1 \end{pmatrix},\qquad C + G \text{ is undefined,}\qquad 3E + 2H = \begin{pmatrix} -6 & 0 & 0 \\ 0 & 15 & 0 \\ 0 & 6 & -8 \end{pmatrix}.$$